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ARITHMETIC

METHODS AND REVIEWS

BY

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AND

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PREFACE

This manual has been written especially to meet the needs of normal students. Incidentally it may be found useful to other students and teachers. Besides methods, it presents reviews of subject matter. While the teacher who has not had methods is greatly handicapped, the theory of teaching is of little value to one who has not mastered the thing to be taught.

The letters of Mrs. Hornbrook are specifically for the student teachers of the Training School. The first sets forth the general scheme of her work and gives a number of illustrative exercises and games. The second gives a general outline of the course, which will be fully amplified in the supervisor's meetings from week to week.

The chapter on drills gives suggestive exercises intended to help the busy teacher and to keep the pupil occupied in what will interest him and cause him to know.

The chapter on course gives a skeleton outline for work beginning with the fifth year. It is tentative, suggesting an order of sequence.

A large percentage of normal students have had no arithmetic since leaving the grammar grades, hence their knowledge of the subject is quite limited and their work more or less mechanical. They need a firmer grasp of the subject matter and the feeling of security which comes with conscious mastery. The chapter on reviews is designed to supply these needs. The arrangement is strictly topical. While there are many simple exercises and problems the work is in general too difficult for grammar grade pupils. The teacher, however, should know more than she is expected to teach. The many model solutions are intended to furnish good forms, which is important, and incidentally to illustrate the best ways of doing the work.

Modestly yet hopefully this manual is offered to the aspiring, earnest teacher. The more ambitious task of presenting the psychology and pedagogy of the subject is left for others.

W. H. B.

State Normal, San Jose,
September, 1914.

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I.

INTRODUCTION

AIM.

The manner of teaching arithmetic and its content are determined in a large measure by the purpose for which it is taught and the teacher's conception of its educational value. If it is taught merely as a tool subject, only such topics will be considered as are of direct service and the method will be such as will bring about the mastery of these topics with the least possible expenditure of time and energy. On the other hand, if only the culture value of the subject is taken into consideration, the method of study and the way in which it is presented will be deemed of more importance than the topics themselves.

Arithmetic is a tool subject, and this fact makes it imperative that certain topics be included in the course, and taught with that degree of thoroughness that will enable the pupil to do the work readily and accurately. The form of work should be such as to enable the learner to get the desired result as quickly and correctly as is consistent with clearness.

Arithmetic as a tool is used by the pupil in pursuing his studies in the elementary schools. For this purpose he needs to understand the fundamental processes as applied to whole numbers, common fractions, and decimals. In the secondary school arithmetic is needed in the mathematical work in chemistry and in physics. For these the pupil needs a thorough grounding in common fractions, decimals, and percentage. Business life demands a knowledge of percentage, interest, mensuration, and lumber measure in varying degrees according to one's occupation. Since arithmetic is seldom studied outside of the elementary school these topics should be included in the course.

Business men have devised certain self explanatory forms and direct ways of reaching results which should be taught when the proper time comes, instead of the cumbersome and mechanical forms usually given.

School time is growing time mentally as well as physically. Whatever the child studies should minister to this growth, each subject or activity in its own peculiar way. Arithmetic as a culture subject ought to be instrumental in the formation of certain good habits. It ought to train to a neat, methodical arrangement of material, a careful and thoughtful study of the meaning of expressions, and a disposition to inquire for reasons and search for truth. Arithmetic ought also to train in thinking and in clear and concise expression of relations, not by the repetition of set formulas, but by the logical sequence.

Arithmetic is valuable as a medium through which the pupil gets information. Customs and forms prevailing in the business world may and should be learned. These will be recognized as of vital importance and will be lastingly impressed if presented in living practical problems. Thru problems the child can get a saner knowledge of the commercial and geographic importance of nations and of the problems of civilization.

MENTAL ATTITUDE.

Authority—Language looks to usage past and present for its rules. The

historian looks to the records and traditions of the past and sifts truth from them. "I have looked it up and I find it so", is the final statement of linguist or historian.

Respect for authority is an important attitude of mind. It urges man to stop and consider carefully before discarding present laws and customs and substituting untried theories in their stead.

Experiment—The scientist is a questioner, a doubter. He is not satisfied with present conclusions. Authority is to him a finger pointing in a direction in which truth may be sought, and he seeks to verify or disprove by his own investigation. "I have made careful examination and I find it so", is his tentative statement.

To this modern scientific spirit much of the world's progress is due.

Proof—The element of necessity underlies the progress of the mathematician. Given certain axioms and postulates, his conclusions follow inevitably. He may examine his work for error of statement or oversight, but finding none he can not question the result. When a truth has been settled mathematically there is an end of the controversy. "I have demonstrated therefore I know", is final with the mathematician. He can turn his attention to other problems.

This feeling of certainty can and should be in the mind of the student of mathematics from the beginning.

INTEREST-MOTIVE.

The educational value of a study depends largely on the amount of mental energy called forth by it, and this is dependent on the interest with which the study is pursued and the motive which awakens this interest. Teacher and pupil must co-operate if there is to be substantial progress in education. A task may be assigned and the pupil required to perfrom it, incited by fear of punishment or hope of reward. Thru such means the pupil may acquire knowledge and skill, but he will have little pleasure in his task, and the resultant mental growth will be a minimum.

Use.—Use is commonly considered the prime motive power. If this use is in the future its influence on the average elementary pupil is slight. The child lives in the present and for the present, and the incentive of the future usefulness of a study supplies only an artificial stimulus. Present use arises from present necessity, and can last only while the necessity exists, hence its influence is at best but temporary. It emphasizes the necessity for certain knowledge and serves as a reason for the study of a particular topic, but it does not and can not furnish the motive for the persistent study which is necessary if there is to be any permanent beneficial educational result.

Inherent Interest—The motive which will call forth persistent interested effort is that which comes from the subject itself as influenced by the teacher and the manner of presentation. To know, to understand, to be able to do, to be skilled in doing bring their own reward and awaken an abiding and increasing interest. Ken, can, and king are words closely related in language and in human nature. He who kens can, and he who can is made king; and rejoicing in his power he seeks to increase it.

Arithmetic properly studied affords its own incentive and reward. To know numbers and to be able to handle them with accuracy and intelligence give genuine pleasure to the student; there is a definiteness about the results and a certainty about the conclusions which give satisfaction; and one feels that it is worth while to pursue a subject in which there is a consciousness of increasing power.

Freed from drudgery and grind, it is possible for arithmetic to be made from the first a pleasant and interesting, therefore profitable study.

Play—The recreative value of play has long been recognized. Cities are realizing that play may be made to have a socializing and civilizing force, and are spending millions in purchasing and equipping grounds and providing them with proper supervision. Play is no less play because it is wisely directed. We have been strangely slow in recognizing the educational value of play, overlooking the fact that a large part of our knowledge and skill have been attained thru that channel. The boy, and sometimes the girl, learns to ride, to hunt, swim, row, and skate thru play The farm boy learns to handle tools, drive, plow, and do a hundred other things, thru play, and while learning enjoys himself to the full. The eagerness with which children pursue manual training and domestic science is due largely to the gratification of the play instinct. Children do not play at a thing because it is easy. From the child trundling his wagon to the football enthusiast, real play is real work gladly and rigorously performed. The play spirit enables one to do his best under the best possible conditions. The child's play is his work. Happy is the adult whose work becomes his play.

The play instinct should be largely utilized in teaching arithmetic. The child's desire to count and his joy in counting, his pleasure in constructing and destroying that he may construct again, his inquisitiveness and eagerness in making his own discoveries should be made to contribute to his advancement, to lay the foundation for future discoveries, and to awaken an abiding interest. Thruout the course the arithmetic recitation should be looked forward to with pleasure and keen anticipation, and should leave a memory of happy attainment and a determination for continued effort.

PHASES.

In considering the subject, one must recognize two distinct phases of the work. These are: first, the ability to handle numbers, i. e., to perform the operations, addition, subtraction, multiplication, and division of whole numbers, common fractions, and decimals, and extracting roots, and, second, the application of these operations in concrete situations. Tho usually taught together, these phases are logically distinct, and methods which will develop efficiency in one will in no wise advance the other. The ability to determine what operation should be applied in a given problem does not come with the ability to perform the operation, neither is it learned thru a set form of words. It is a matter of experience and judgment. Within the horizon of the child's experience, and especially within the circle of his interests, he seldom makes a mistake. It is when we place before him situations foreign to his experience or in which he has no personal interest that he flounders about. Because of these facts no textbook in arithmetic should be placed in the hands of the pupils during the first three years. During the first year he should wrestle entirely with problems which arise out of his work or play. It is not necessary that these problems involve number. The laying out of a flower bed, the building of a cardboard house, or the putting together of spools and box to make a wagon will result in growth and power. The weights and measures in common use should be handled that they may furnish material for future work. Problems for the second and third years should arise from the work or play or come from the living teacher and should have direct reference to the child's immediate environment. In later problem work, attention should first be directed to the meaning of the problem, what is given and what is required. Care should be taken, especially when taking up a new type of problem, that small numbers shall be used and that the operations involved be not complex. Forms of analysis are of questionable value.

Rationalization—Shall the operations be taught as a matter of pure memory and drill, or shall there be an appeal to the understanding? Whatever method is adopted there must be much resort to drill and much memory work must be required. The pupil must finally come to perform the work more or less automatically; accuracy and a greater or less degree of speed must become habitual. If, however, memory work alone is resorted to the result will be that the pupil will become mechanical in the extreme. Mechanical in performing his operations, the child will almost certainly become a guesser as to what operation should be performed. Too many will be so under the best conditions. On the other hand many pedagogical sins have been committed and much harm has been done thru attempts at rationalization. Authors and teachers alike have erred in this respect; bad forms have been impressed and bad habits have been encouraged. These forms being placed before the child when he is learning a process impress themselves upon him and they become to him a necessary part of the operation. Even the use of splints, beans, counters, and the like, in rationalization, may become stumbling blocks. Some pupils become slaves to objects, and can work intelligently only so far as they can see or image the object, and when such objective work is impossible because of the largeness of the numbers fail utterly or become the worst of mechanical workers. Others resort to marks on boards or papers, count the fingers, or make motions in the air. A few survive and become fairly good mathematicians despite bad methods. In the following pages the attempt is made to bring about an intelligent appreciation of the processes and yet avoid the evils referred to, for only by so doing can the work be made enjoyable. What is learned as a distasteful task has little value educationally. It must not be expected that any pupil can understand all the processes as their use becomes necessary. The rationale of some operations is too difficult for children at the age at which such operations should be learned. In other cases it is of little consequence. The inversion of the divisor in division of fractions is an instance of the first kind, why to begin at the left in division is of the second. Many pupils will get little good out of any attempt at rationalization. This fact, however, should not keep the teacher from making any such attempt. Some seeds may rot in the soil, others may be long in germination, but for the sake of those which grow we must not withhold the planting. Mr. McMurray's rule is not a bad one, tho not without exceptions: "It is folly to sacrifice the present for the future. Any subject that cannot be fairly comprehended at the time it is presented should be excluded" (Ed. Rev., Vol. 27, p. 482). While, then, we must not attempt complete rationalization, whenever the child of ordinary ability can be led to comprehend the reason for an operation, he should be given the opportunity.

SPECIAL FEATURES.

In a few respects the course herein outlined differs from others.

Exposure-Play—In the work outlined for the primary grades Mrs. Hornbrook has shown the happy way in which play may be made to count toward real acquisition in arithmetic. She has outlined a number of interesting exercises, and these will suggest many others to the resourceful teacher. These exercises build up and keep continually before the child a useful picture of numbers to one hundred, the full value of which can be appreciated only by the teacher who has patiently and thoughtfully used it. They bring the child into the presence of laws thinly veiled, and the pupils soon begin to find out for themselves facts about numbers. Thru these exposures and thru play the subconscious mind is charged. By what alchemy it works we do not know, but we know that thru it the child finds pleasure and profit in handling numbers.

Law—An attempt is made to lead the child to investigate for himself and thru such investigation to discover truth. The pleasure of personal discovery is great and the habit of looking for reasons and laws is a good educational asset. There are many useful laws governing numbers so simple that the child can

grasp them. These are utilized and emphasized. Children take delight in certain exercises, such as counting by twos, tens or fives. These pleasurable exercises are made to do service in learning the combinations of these numbers in addition, multiplication, etc.

Groups—The pupils are encouraged to recognize certain group results such as the groups making eight, nine, ten, and so on. These groups are emphasized at the proper time and the habit of adding them as a single number is formed. Groups making nine should be learned after the child learns to add nines; in general a group making a given number should be learned when the child has learned to add that number.

Fixing—The pupils are not required or expected to use a combination in a miscellaneous way before it has been well fixed in the memory. To do so is likely to result in bad habits and inaccuracies. To fix a combination requires much repetition and it ought to be interested repetition. This is secured thru counting, decade work, and simple column work, making use of the eye, the ear, and the sense of rhythm.

Small Numbers—In the work of the first three years no attempt is made to deal with large numbers. The pupil is taught to read and write numbers to 10,000, but he has no occasion to use large numbers at this age and the mechanical handling of them is postponed. After the child has grown into a knowledge of the principle of place value he is less likely to become a machine. Many children take a delight in reading and writing large numbers and in handling them. Such are not discouraged. Their questions are patiently answered and their feats receive due recognition.

Reasons—Rationalization is not sought thru categorical statement of a general principle. However apt such a statement may be it is likely to be another's rather than the pupil's reason. A method of presentation is sought which will be its own explanation, to be comprehended today or next year.

Inspection—Much use is made of inspection work. It is no longer necessary to find factors, divisors, and multiples of large numbers, and for this reason mechanical methods of finding such factors, etc., are discarded, and in their stead are given methods that will reinforce the power of inspection. Thruout the course such methods are used as will contribute to the mastery of number.

When possible a topic, e. g., common divisors, is introduced for the first time in connection with its use, and is then dealt with only sufficiently to serve the purpose then at hand. It is later taken up by itself and given full statement.

Efficiency—That method of teaching is most efficient which accomplishes the desired result with the best expenditure of time and effort on the part of the pupil and teacher, without at the same time doing violence to the child's intelligence or will. Arithmetic may be learned as an assigned task accompanied by the requisite amount of drill, but such a method will produce mechanical workers and haters of arithmetic. Time may be wasted on useless games and exercises which do not contribute to real advancement. Both extremes should be avoided. Interesting plays, games and exercises which count in the final equipment may be found. Such have been sought in the following pages. No game has been recommended merely because it is pleasing, and the exercises being intelligent and purposeful appeal to the child's interested effort. Crutches should be avoided, and forms for work and ways of doing it which will later be discarded should not be introduced. An exception to the last statement is permissible when the form finally adopted is an abbreviation of the fuller form which should be used at first: e. g., the full and the abbreviated forms for addition and subtraction of fractions.

Efficiency demands that the operations be performed in the most direct manner, the method which leads to the mastery of number being preferred. Inspection work should be encouraged and labor saving devices should be taught in such a manner as to challenge the pupil's best efforts. Actual business forms and business customs should be taught and followed.

Thoroness should be the final goal. The processes that need to be learned are not so numerous that they cannot be mastered, and the feeling of security is very satisfying. There are too many boys and girls who are afraid when they encounter a problem involving fractions, and who surrender when the operation involves the handling of a complex decimal.

In most business operations it is true that only small fractions are used, and that the business man is satisfied if the final result is correct to two or three decimal places. It is also true that science and business are seeking greater accuracy in small things. The astronomer measures time and angles to a hundredth of a second, the axle of the automobile is measured to a thousandth of an inch, and the price of electricity is quoted in hundred thousandths of a cent. "Now a merchant needs astronomy to see them (the profits), and when he locates them they are out some where near the fifth decimal place". The final result can be accurate to two decimal places only when the successive steps have been kept well in hand. A result may be more accurate or less accurate than the data according as the error has been multiplied or divided. The pupil should be trained to discriminate.

It will be seen that the aim of all our work is to present the subject matter in such a way as to meet the psychological conditions of the learners. As the natural aptitudes and the environmental conditions of individual children vary greatly, it is evident that in order to reach efficiency there must be a sectioning of the grades into groups, by which the quick and successful children are given freedom to advance at their own rate while the slow or unsuccessful pupils are allowed to carry on their work in the way natural to them. It is also important that no thought of inferiority or superiority in the work of any of the groups should be given to the children or held by their teachers. There can be no efficient instruction unless the inherent powers and natural rhythm of the children are considered in the teaching effort. The working plans of the Training School are such as to secure this division of the grades into small sections at varying stages of progress. This makes possible that adaptation of the work to individual needs and powers without which the most careful methods of presentation do not insure success.

This idea of definitely and frankly adjusting the work to the individual appears again and again in the early pages of the manual and is assumed thruout the book.

AN OPEN LETTER

From a Supervisor of Arithmetic to the Students Teaching Arithmetic in

THE TRAINING DEPARTMENT

of

THE STATE NORMAL SCHOOL AT SAN JOSE, CAL.

DEAR FRIENDS:

Altho not all of you can be so fortunate as to be assigned to the teaching of the little children who begin number work, you will see at once that it is important that every one of you shall have a clear understanding of the aims, the principles and the processes of these beginnings as well as those of the later work. This is necessary in order that the work of each of you during your twelve weeks of practice may be rightly related to that of others.

In these few weeks you are to begin your professional study of children's minds in their reactions upon the truths of mathematics. Your success in teaching will depend: first, upon your habit of close, intelligent observation of these reactions; second, upon your skill in interpreting what you observe; and third, upon your ability to present the facts of mathematics in ways suitable to the minds of the children as you find them to be. This letter is written to help you. It refers to the work of the first four grades.

There are two ideas of number, viewed in the light of child psychology, which underlie the plans of teaching here presented. These are the "number series idea" and the "number form idea." If you are to use the plans intelligently you will need to understand these basic principles. In Dr. Stanley Hall's Educational Problems, Vol. 2, pp. 350-356, you will find an excellent statement of the Series Idea in Number, and many known facts about Number Forms, with a short, encouraging reference to the application of these ideas in the practical plans, some of which are given in this letter.

The Series Idea In Number The number series idea was brought to the notice of American educators by an article in the Pedagogical Seminary for October, 1897, written by Dr. D. E. Phillips, then of Clark University. It is now generally accepted by writers on primary arithmetic and is practically applied in most modern textbooks of that subject.

Very briefly stated, the idea is this;—

Ordinary children in their early years think of numbers as a series of sounds, "one, two, three," etc. They like to bring this series into their consciousness and play with it. They repeat these number words, generally attaching no more meaning to them than to "eeny, meeny, miny mo," but enjoying the rhythms, the repetitions, and the jingle of this sound series. They love to show to grown-ups their new and interesting accomplishment of counting.

Your first work will be to find out how well your individual children can count and then to help them to count perfectly to 100. This is as pleasant to them as any play. There may be a little stiffness or shyness at first, but that will disappear as you get into happy, sympathetic relations with the little people.

Query. Can you recall any instances in your school life in which your success in learning was affected, either favorably or unfavorably by your feelings toward your teacher or hers toward you?

Note. The queries scattered thru this letter are to be answered by you at the weekly conferences.

Number Forms The discovery of the existence in many minds of certain definite visualizations called "number forms" was given to the world in 1883 by Sir Francis Galton, in the book, Inquiry into Human Faculty. It has been confirmed by many later investigators.

Stated in very condensed form, the facts are these:—

Many children in their early gropings among numbers and figures, make a mental picture of the number series, usually up to 100, sometimes far beyond. They visualize the number symbols, 1, 2, 3, etc. as a succession of figures. They see them mentally at definite distances and directions from one another as if on a printed page. The lines of figures thus formed in the mental picture are sometimes straight or broken, sometimes curved, sometimes in spirals, sometimes in parallels. They differ in different minds. It is estimated by psychologists that about five per cent of adults retain and use the number forms that they built up in childhood, as a help in working with numbers. Sometimes the forms are very complicated, twisted and irregular like one that was given to me last year by a Junior student. Altho it seemed a very inconvenient form to use, she assured me that it was a constant help to her in reckoning. Of the many hundreds of number forms that have been reported, mostly irregular, only one was complained of by its possessor as being "troublesome" on account of the bending and doubling of its lines. But certainly the possessor of an even, regular number form like that given to me by a teacher in our school last year is fortunate. You will find in the school library a most interesting and instructive article upon this subject,—The Genesis of Number Forms,—by Dr. D. E. Phillips, in the American Journal of Psychology, Vol. 8, No. 4.

The facts concerning these spontaneous visualizations of the number series have, as Pres. Butler of Columbia University remarked editorially (Educational Review, May, 1893), "a most direct bearing upon the teaching of elementary arithmetic." They are of great practical importance to us, for instead of allowing our pupils to form irregular, inconvenient mental diagrams of the number series, or none at all, we are going to give to each child by means of charts and other apparatus an opportunity to use freely a visible, tangible representation of the series up to 100 in a regular unchanging form. With this he can make his own discoveries of the facts of number, or can readily perceive the number facts pointed out by his teachers and classmates.

This is not a new, untried project. Plans based upon this idea I began to work out in 1886, presenting some of them in an educational magazine in 1893 and in a textbook in 1898. For many years I have had the pleasure of knowing that other educators were working with the same thought. In the recent writings of some leading California educators the use of charts similar to ours is urged. One of them is given in our state textbook of primary arithmetic (p. 17). Some happy results were obtained in our school last year by the student teachers of one grade who used some of these plans quite successfully with a short period of supervisory help. We are expecting fine results from your work this year, and we are planning to give you all the supervisory, informational help you need,—and not a bit more. We want you to be independent, alert, fertile-minded workers.

This year (1913) there will be three or four grades beginning number work from the 3 B down. The work here described is planned for first and second grade pupils, and is made suitable for children thru the 3 B grade by simply allowing them to advance more rapidly.

The First Lessons In the first lessons the number chart of fives, as given below * is before the class.

1	11	21	31	41	51	61	71	81	91
2	12	22	32	42	52	62	72	82	92
3	13	23	33	43	53	63	73	83	93
4	14	24	34	44	54	64	74	84	94
5	**15**	**25**	**35**	**45**	**55**	**65**	**75**	**85**	**95**
6	16	26	36	46	56	66	76	86	96
7	17	27	37	47	57	67	77	87	97
8	18	28	38	48	58	68	78	88	98
9	19	29	39	49	59	69	79	89	99
10	**20**	**30**	**40**	**50**	**60**	**70**	**80**	**90**	**100**

The chart is copied on the board in large, plain figures. The multiples of five are written in larger figures than the other numbers with crayon of some light color, never in a dull color.

Queries. Why should these figures be large and bright? Can you give any psychological fact that suggests a reason?

There are three different things which the child must master in his early work. They must not be confused. They are:

1st. **The sound series up to 100.**

2nd. **The sight series up to 100.**

3rd. **The use of these series in applying number to objects.**

It will take *many weeks* for the little ones to master these three basic elements. For those below the 3 B grade fifteen or twenty minutes a day are allowed; for the 3 B and those above, forty minutes. Each group will advance at its own rate, doing what it can from day to day without hurry or worry. We shall use many different exercises and simple childish plays. Children naturally love number and they love to play. We will combine these two natural interests and as a result will obtain clear perceptions of number, and the joy which children find in doing something worth while in the company of their mates.

Organized, purposeful play, guided by the teacher, leads to voluntary, resultful work by the pupils.

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*The charts in this pamphlet are taken from Hornbrook's Primary Arithmetic by permission of the American Book Co.

Learning The Sound Series Most children beginning number work know a part of the sound series. They can count a little way with more or less correctness. So we shall begin by using this knowledge as a basis, passing gradually "from the known to the unknown."

There are many good ways of beginning the study of number. I have selected one of them and will describe some typical work of a teacher, whom we will call Miss Smith and will image as a young woman of charming personality, working with one of our Training School groups of about a dozen children. This work is not given as a model to be copied, but as a series of concrete illustrations of principles to be interpreted. Thoughtfully read, it will give you mental pictures of schoolroom activities differing probably from those in which you figured as a child. Educational thought has changed very rapidly in the last few years, and you have come to the San Jose Normal to get the thought of the present, not of the past. Your previous training in the department of Psychology, reinforced by special reading, will enable you to understand the psychological principles upon which the plans are based.

It is the first lesson. Miss Smith begins in the usual way by talking with the children about counting. One child thinks he can count to 30, another believes that he can count "a whole lot" and so on. "Now all count with me," she says, and begins to count slowly. The "counting" is simply giving the number names in their true order. Their significance is not considered at all. That comes later. The children are now getting the sounds in their true sequence. As the counting goes on, the teacher makes mental notes of the pupils who fall by the wayside, and of those who go on triumphantly until she gives the signal to stop. The signal is given as soon as she sees signs of failing powers or flagging interest.

In this first lesson the young teacher begins her study of the minds of the children whom she is to teach. "What can this child do and how does he do it, and how can I best help him to do the next thing?" are frequently recurring thoughts.

In the ideal conditions of your work,—with your groups of about a dozen children, each in its own pleasant little classroom, with sympathetic supervisory help close at hand and with the children looking to you as the bringer of something new and interesting, it will be only a short time until, if you use intelligent effort, you will be able to see your whole group as individuals, each with his own abilities and temporary inabilities, which you are to help him remove. Then the true interchange of thought between teacher and pupil will take place. You will watch and assist the development of the mathematical sense of each child. He will learn to look upon you as his own personal helper. The more clearly he sees you in that light, the more trustfully and happily he will follow your guidance. When this beautiful intimacy of thought is established, the need for discipline—in the old harsh sense—disappears. Instead we shall have obedience, pleasure, and clear understanding.

Because the presence of outsiders breaks the flow of thought between pupils and teachers, you and your children are to be shielded as much as possible from outside visitors. The parents of the pupils are more than welcome. We are anxious to consult them. They can help us to understand the children. It will be well to have it understood that when, for instance, Mary's mother or father comes, Mary shall do an unusual amount of reciting in order that they may judge of her attainments or defficiencies. And so with each child. Parents who have entrusted their children to our care are in sympathy with us in our work. They have a right to know of their children's success and of their temporary failures, if they care to follow the course of instruction closely. Altho we earnestly desire that they shall not break that course by attempting to teach their children, we are anxious to learn from them all we can about the ways in which the minds of our pupils react upon the work.

Let us return to our ideal teacher whom we left watching her little ones as they counted.

After the group counting, which has served to break the ice, the individual activities begin, the voluntary work or "stunts" as they are called. "Who wants to count all alone for us?" says Miss Smith. Volunteers are ready with the usual signal. Tom is chosen. Glad of his chance to display his accomplishments, he begins to rattle off the numbers.

A point of importance suggests itself here.

Perhaps Tom blunders, puts 19 directly after 15. As soon as the mistake is corrected either by himself or by some one designated by the teacher, he must stop. Perfection is the standard and nothing short of it is acceptable. The slightest error "spoils the stunt." This rule is to be invariable and is explained at the first lesson in an easy, pleasant way as "the way we play the game." So Tom sits down encouraged by a word or a look from the teacher to feel that he will "soon do it all right." Mary has the floor, and Tom, instead of stumbling along in a maze of uncertainty with half grasped, embarrassing corrections from his teacher, hears Mary's quiet smooth counting. If he is an alert, sensitive child, he follows Mary's performance attentively with a keen desire to equal it. If he is slow, unawakened as yet, his attention may be slack, but he is *unconsciously becoming familiarized with the number series.*

Another point. Henry may say with pride, "I can count to 100, and Tom can only count to 15." This is Henry's mistaken way of trying to win the approval of his teacher and classmates. It is contrary to the spirit of respect and consideration for others which is to rule. The teacher may say, "Can you?" adding in a low, confidential tone with a very significant look, "But you mustn't say anything about it." Or she may answer in a light, easy way, "Well, if that is so we will have to help him, won't we?"

Query. To what feeling does each of these answers appeal?

The oral counting is continued from day to day, along with the chart work and the counting of objects which I shall describe later. After a time Miss Smith introduces decade counting. "We will count by 'parts' now," she says. "Who can begin at 21 and count?" Later as a little anticipation of the process of adding numbers to 20, 30, etc., she asks, "Who can begin at 41 and count five," or "Who is ready to begin at 71 and count ten?" Not quite knowing her pupils yet, she sometimes suggests feats of ability that prove to be beyond them. In that case she quickly and smilingly substitutes something else. She does not invite failure and confused, unhappy thinking by forcing upon her pupils anything for which she sees that they are not yet ready.

One of the wisest things written lately about education is the following from a little book;—The Montessori System—by Dr. Theodate Smith, of Clark University. "Failure is a negation showing that the child is not yet ready for that particular exercise," p. vii.

A child may advantageously see work done by his mates that he does not yet understand. He will not be confused by it unless he himself is called upon for it. Hence the value of the principle of "voluntary effort" used so much in our school and in other schools of the modern type. Calling for voluntary work the teacher is careful to see that each child has the opportunity to express his thought, and she cares particularly for the weak little ones who need her help before they are able to present the results of straight thinking.

It is to be observed that, altho much freedom is allowed, Miss Smith's classroom is not ruled by the children's whims. She knows that she is wiser than her pupils and that upon her rests the responsibility of the work, that the children are there to learn and that they must not be hindered by the chaotic conditions which would arise without a competent person in control. She is the gentle, calm, confident director of affairs whose directions are to be obeyed. That is what we expect you to be in your classrooms. The principal of the Training School and every one of your supervisors will give you all needed support in that position.

Learning The Sight Series At first the children see the chart as a mixed-up blurry picture of meaningless lines and surfaces. But the series of number words which they have learned is the key that will unlock for them all its values.

After the oral counting in the first day's lesson, Miss Smith begins the chart work that is to lead to the knowledge of the sight series, and to the realization of many of the relations of numbers. Calling attention to the chart she says, "All the numbers that you have been saying are on this chart, and I am going to show you how to find them." (With her little ones she does not make the distinction between "numbers" and "figures" that exact mathematical phrasing would demand, nor can we.) She begins to name in order very deliberately the numbers in the first column, pointing to each as she names it. She soon stops and calls on the group to count the numbers while she points. Then she asks, "Who can point while we count?" She chooses Louise, whose power of attention is weak. The child is glad to take the pointer, but she is unable to keep in touch with the counting of the group, and soon has to turn the pointer over to some one else. Miss Smith does not show any sign of disapproval either by word or look, nor does she feel any disapproval. She makes it easy for the child, just as she would for a guest. This is the custom in her classroom.

The ideals of courtesy, when realized in the schoolroom, prevent children from expressing contempt or other unpleasant feelings toward their mates. And in this training school of ours, in which you and I have the sacred responsibility of influencing the dawning thoughts of children and in which this described work of our imagined Miss Smith is really the suggested, prophetic outline of your own work, there are not only ideals of courtesy and good form but there are ideals still higher. We have all heard them beautifully expressed by the principal of the school and by other teachers—the ideals of love and mutual helpfulness and social service. These ideals are to be found not only in our Normal School but in many schools thruout the country. They are a part of the best educational thinking of the present day. They are displacing the selfish, individualistic thought of the old rod-ruled schools.

Miss Smith is trying to hold up these higher ideals, trying to do it in a quiet, non-preaching way by her own attitude towards the failing ones, and by her decisions in the little social questions that arise in the classroom. As far as they are realized these nobler ideals prevent the children not only from expressing, but from *having* unpleasant thoughts about their neighbors.

In the case of Louise the teacher knows that blame, reinforced by the scorn of the class, would be positively harmful to the child, and so she protects her Louise can see that she has failed in this matter. Sooner or later, unless she is subnormal, the desire to equal her mates and the interest in learning will lead her to give voluntary attention. In the meantime while the power of attention is strengthening, Miss Smith tries to be helpful by giving the child frequent opportunities to take part in the exercises under favorable conditions.

In this description you will observe that I do not fix the limit of a day's lesson, but simply give the exercises in a continuous story, in the order in which they follow one another. Of course only one new exercise should be given at a lesson. The amount of work and the number of different exercises to be used each day, depend upon the way in which the children respond to the work and upon the teacher's judgment of it at the time. No grade limits are given here. Each student teacher will take up the work at the beginning of a term where her predecessor has left it.

Miss Smith starts the "stunts" in the learning of the sight series by asking, "Who wants to count and point all alone?" George succeeds in counting to the end of the column. "Stop at the 10," says the teacher. None are allowed to go beyond 10 that day nor for many days.

Query. Why should the pupils not be allowed to go beyond the first ten numbers at this time?

A person entering the room might suppose that the children were reading the figures, but generally they are as yet only finding the names of the figures by counting. As the next step Miss Smith points to the figure 3 and turning to one of her quickest pupils says: "Ruth, what number is this?" "3," replies Ruth. "How did you find out?" Ruth takes the pointer and explains, "I began up here with the 1, and I just counted right down and when

I said 3 that was it." Thus taught and helped by reinforcing suggestions from the teacher, others name figures. It is explained to them that they must take all the time they need, and be very careful. If they hurry or are careless they may make a mistake, which would be a very bad thing indeed.

On another day after they have "named numbers" for a short time they begin to "hunt numbers."

In the previous exercise a figure was pointed out and the pupils had to give its name. In this "hunting" game the name is given and the child must find the figure. These two reverse ways of relating the number symbol and its name are used by Miss Smith in all the little, childish plays by means of which she leads to the easy, accurate reading of numbers. These forms of play, like the oral counting and the chart counting, are given on successive days until no longer needed.

"Look at the chart and find 4," says Miss Smith. They look as directed and count silently, some making gestures towards the chart until they come to "4," and are ready to point out the figure. Unwilling that the class shall hear an error, the teacher is careful to call at first upon the most apt pupils. When the exercise is understood she gives a new form to it by asking, "Who knows a number for the class to find?" Helen's hand is raised. "Come and whisper it to me," says Miss Smith. Helen comes forward and whispers, "5," "Give it to the class," is the direction. When it has been found Helen looks shyly up at Miss Smith and says, "I know where another number is." She is allowed to give it to the class and they find it.

A child who comes forward and directs the group work is called a "leader." Nearly every child loves to "play leader." Of course a blunder robs him of that dignity at once, but it is often necessary for a pupil to give up leadership while his work is still perfect in order to give the others a chance. These are called "true leaders." The teacher keeps a list of the true leaders. Later they will be allowed to do special work.

At this stage the children generally are not yet able to recognize the figures at sight. They are learning to see them as separate things in the chart picture and the next step is to learn their forms. Miss Smith has a set of figures, six or seven inches high, mounted on pasteboard. (This September I am going to lend you a set until you can get them from calendars). She sits before the class, holds up the 1 and running her finger down its length, says, "Who wants to come up and go over this nice big 1 just as I do?" Every one except a very slow, shy little fellow is eager to come. After several have "gone over" the big 1, moving a little bunch of fingers over it without touching it, she turns to the shy child and in an easy, off-hand way calls upon him to come and try. If he still shrinks she excuses him for the present, saying to herself, "I must look out for that little chap. He will need special attention."

In our school where the principle of voluntary effort is applied in so many delightful ways, some of which you can see any day by spending the assembly period in one of the school halls of assembly, it is unusual to find a child who would refuse such an opportunity as Miss Smith offers. Such a child needs to have his work skillfully adapted to his capacity. Gradually under the sunshine of the teacher's kind thought in the genial atmosphere of a classroom in which consideration for others is the ideal, the little one's fear and reserve will melt away. He will begin to work. And then how happy he will be when he finds himself doing the things that the others are doing, those puzzling, embarrassing things that he had thought too hard for him even to try.

The big 1 having been traced, Miss Smith holds up the 2. This is more difficult. Some start at the wrong place or go in the wrong way. The small hands have to be guided. "You must always begin at the nose of the 2," said a little girl, "and go right round to the tail; and you mustn't ever rub the fur the wrong way."

In the comments which our pupils are encouraged to make, subject to the laws of courtesy and good sense, they show to us the workings of their minds and they often give useful suggestions. A little boy in one of my experimental classes last year called the big 2, which I gave him for a copy, a "papa-2," and proceeded to make a smaller one which he called a "mamma-2," and one still smaller which he called a "little-2," This interested other pupils and one of them added a tiny little figure which he called a "baby-2." We allow the pupils to express their childish fancies but do not impose them upon other children to whom they do not appeal.

Miss Smith does not introduce any more figures that day because she is not willing to risk confusing her pupils by presenting too many forms at one lesson.

Queries. When several strangers are introduced to you at one time do you ever have difficulty in applying the right names to them afterwards? You have an adult brain.. How about the brains of little children in comparison with yours?

Tomorrow Miss Smith will have big 2's written on the board, one for each child to trace. Those who do it right will be allowed to take a crayon and trace the figure. Then they will copy it. Arm movement "with a good big swing" is the ideal she will set before them. When 2 is so well learned that it can be recognized anywhere she will present the 3 in the same way, then the other digits in their order. She never presents more than one digit at a lesson. When 4 is reached, Miss Smith does not require the pupils to trace the printed figures which she shows. Instead she tells them, "When we write the 4 we always leave the top open." Then she sets them to tracing and copying the large 4's, that she has written on the board. She knows that children are apt to turn the 6 the wrong way, so she gives them much practice on that figure, insisting upon the arm movement.

An interesting psychological explanation of this tendency to reverse figures and drawings has been given and we will discuss it at one of our weekly conferences. At any rate we will prevent our pupils from reversing their figures if we can.

There are two ways of tracing the 8, Miss Smith insists upon the movement used by the supervisor of writing.

Each day before taking up the new figure the pupils write those already learned in a column like the first column of the chart. It pleases them to see their columns grow as the new figures are added, and their interest grows with it. If, however, any member of the group blunders in recognizing or copying a figure already given no new figure will be given that day. Instead some other exercise will be used.

With regard to this matter of diverting attention from errors you will find one thing which may seem to you at first to be strange, altho experienced teachers know it well and profit by it. When your pupils get wrong ideas and their little brains are "all muddled up," if you labor with them showing them the right thing in contrast with the wrong, you will only make matters worse. All your strenuous efforts are worse than wasted. Instead of laboring, the skilful teacher simply shows the right, briefly and easily, and then turns to something else. On another day she leads carefully up to that upon which the pupils were confused. The chances are that in the interval certain little modifications of their brains have taken place as the result of her work and of the children's desire to know its meaning and that because of this interval the subject is easily made clear to them. If not, it is evident that they are "not yet ready" for that particular exercise.

Query. Why is it in the long run a saving of time and effort to present the digits so slowly and carefully?

Many other exercises are going on in Miss Smith's room from day to day by which the children are learning the sight series and all unconsciously getting the first dim ideas of the relations of numbers, *ideas that will brighten into consciousness later.*

In one of the first lessons Miss Smith supplied each child with a chart for his own personal use. Soon she will have her pupils make the charts for themselves. The chart was made as follows:

Upon a piece of Manila paper one foot square (obtained from the supply rooms) she drew, very lightly, a ten-inch square. She divided it into inch squares by light, almost invisible lines, and wrote the hundred numbers in the squares. The multiples of five are larger than the other numbers and are written in bright colors with crayola. After the children have examined their charts and given all the comments or questions that she thinks worth while, she sets them to counting and pointing out the numbers. Then (probably on the next day) she gives to each child a folded paper containing squares of pasteboard slightly smaller than inch-squares upon which are pasted printed figures cut from calendars or made by rubber stamps. (Written figures may be used, but they should not be written with a pen. They need to be clear and bold. If cotton is wrapped smoothly around the point of a pencil and the pencil is dipped in ink, figures can be made with it almost as clear and bright as print.) Multiples of five are distinguished by color and size. Each paper contains only the first ten numbers. She allows the children to work in pairs if they choose. The thing to be done is to place the pasteboard numbers upon the corresponding chart numbers. To place them where they belong and "right side up with care" is not an easy matter for the little people. It furnishes interesting occupation for many days, a short time each day.

Some day after the ten numbers are placed on the charts in due order, Miss Smith shows the children "how to play with hidden numbers." The first direction is, "Turn the squares over just where they are." "Now the numbers are all hidden out of sight," she says. "Let us see if everybody knows where they are." Stepping to Roy's desk, she puts a finger on a pasteboard square and asks, "What is under this?" "6," he replies "How did you know, Roy?" "I counted them. I know them all. This is 4 and this is 9 and—" She stops him to address the class. "Be ready. Roy and I are coming around to see if you know where the numbers are hidden." "What is it, Walter?" she says to a boy whose hand is raised. "I know them all, too." He quickly designates some of the numbers. "Walter will take this outside row, Roy the other, and I will take the middle rows.

Query. Why is it better for her to take the middle rows?

Another day after the numbers are hidden she starts the game of "letting the numbers out." "We are going to let the numbers out of the places where they are hidden," she says. "Uncover 4." The children count, if necessary, until 4 is reached, pick up the square and lay it over to the left of the chart column. The teacher calls on James to "play leader." He comes forward and designates the numbers to be uncovered, leaving her free to watch individuals. Some, without counting, know the positions of the hidden numbers. There is no chance of a mistake if the child is careful to verify his thought by counting. Under these circumstances a mistake is not to be tolerated. Miss Smith emphasizes the idea strongly beforehand, (not, of course, after the mistake has been made. I hope you will emphasize it in the same way with your pupils.) In spite of the teacher's emphasis upon care and upon "counting till you know," Kate, an irresponsible little creature, sitting in the front row, blunders. (Such things happen right along as you will find). Miss Smith steps over to Kate's desk and says quietly, "Oh, I am so sorry. You will have to stop." (Rule of the game.) "You may watch Anna's work." Anna is a slow, painstaking worker. As Kate sees her uncovering the numbers without a mistake she says to herself, "I can do that." The next time the game is played she does do it.

Queries. Why should a teacher place her unsuccessful pupils in the front row or as far forward as possible? Usually they prefer to sit as far back as possible. Why?

Another exercise, "building ten," is given with the same purpose of helping the pupils to a knowledge of the number symbols and their relative positions. The individual charts are put out of sight. The pupils arrange the little number squares on their desks in the order of the numbers 1—10. If they are uncertain they must refer to the chart. They must not make mistakes when it is so easy to prevent them by simply looking at the chart. When they are able to build the first ten without help from the chart, or from any person, they take up the next decade in the same way. Miss Smith has wrapped the number squares 11—20 in papers of a different color from those which contained the first ten. She does this in order that she may easily distinguish them. She keeps each set of papers in a separate marked envelope and the envelopes in a box in the closet. Before giving out the papers she makes herself sure that the set is perfect. When gathering them up she appoints some one as a committee to see that none are lost. This committeeship is eagerly sought.

The preparation and care of material and apparatus for number play and number work demand much attention to detail. But it is vastly easier to attend to material and apparatus than it is to teach number successfully without such concrete aids to thinking in the early stages of the work.

At the time of this writing there are in Prof. Baker's office a thousand each of tags, test cards and square feet of Manila paper, all cut to size and waiting for you to write clear, bold figures upon them, with your cotton-covered pencil-points or some equally effective substitute. There are also three new tagboards besides my old one. These will be passed around as needed. In Miss McCabe's office there are three photographs on the wall showing children working with apparatus. One picture shows two little boys down on the floor, "building tens" with figures cut from calendars. A large piece of cambric has been laid down for them to work upon. (You can use newspapers if you prefer.) In another picture two others are building the table of fives on the tagboard. In the third a little fellow is building the table of nines on the buttonboard while his companion waits for the game which will follow as soon as it is completed. In Miss Hall's office there is a picture showing also children working with test cards, strips, nailboards, peg boxes, etc. I will show you how to prepare and use any of these that may seem desirable to work with in your group. Such occupations are more useful than class drill in helping the child to form true concepts of the number series, and as they are really a form of quiet, social play, the children enjoy them greatly, and can engage in them without fatigue for a

much longer time than in ordinary class work or in games where the attention of the whole class is demanded. Class exercises, whether work or play, should be brief. Children have not the power of sustained attention, and it is worse than useless to try to hold their attention after their stock of brain power is exhausted, But in the occupations something is to be accomplished; as, for instance, the making of a "pretty chart." The creative joy is aroused, concentration follows, and generally the children are not willing to stop until the thing is finished. For that reason you will find it better to put the occupations in the latter part of the period after the class exercises. In the case of quick pupils for whom a particular class exercise is not necessary, it is well to leave them out of it and give them an occupation, while you work with the slow ones, mind to mind. Your best and brightest pupils are likely to become impishly troublesome if they are held down to the pace of the slow. (Really one can't help thinking that they have justification.) They have a right to advance at their own rate and it is an easy matter for a teacher, whose day's lesson is well planned, to give to the quick pupils an occupation such as written work or chart work or apparatus work in which they are profitably employed, not merely occupied, but really learning. Such children dislike what they call "baby work." "Give us something harder," they often say. "I've got something nice and hard to do today," said a little fellow proudly showing his occupation. Arnold Bennett, an English writer, was astonished to see children in New York "grabbing knowledge from their teachers," as he expressed it. If you are successful in your teaching you will see your pupils "grabbing knowledge" from you and from their mates. The little ones often have sweeter ways of getting knowledge. When a little child comes to you saying, perhaps with a gentle tug at your skirt or a soft touch on your hand, "Please come show me about this," your first disengaged moment is the psychological moment for giving him the knowledge he wants. Perhaps as you watch the little worker you will feel an impulse to put your arms around him and give him a little hug. But if you are wise you will nobly resist the impulse and will keep to the easy, pleasant manner suitable to the schoolroom. However, if you feel that impulse, it is probable that the teaching instinct, so nearly akin to the parental instincts, is welling up in your mind and heart. This instinct for loving, guiding and instructing the young, if reinforced by reason and good judgment, by the study of your pupils and also of educational principles will bring you success and happiness in your work. Then you will not become a tense, nerve-racked schoolma'am or domineering, dreaded schoolmaster. Instead you will be a loving leader of children, guiding them into the realms of the world's knowledge.

I have emphasized the fact that slow pupils should not be prodded by blame. Quick pupils should not be stimulated and make heady by praise. A child is no more deserving of praise for being more capable that his mates than for being taller or handsomer or having a better father and mother than his mates. These are all matters of private congratulation but not of public praise. The fact is that under those plans of teaching by which children were required to commit to memory a certain number of facts in a certain time, whether they understood them or not, praise and blame were stimuli used to force them into that very disagreeable form of activity, to make them "get over the ground." Such stimuli have no place in our scheme of things, where every child is to be given (or allowed to help himself to) all that he can carry at the time without having any dead weight of misconceived facts piled upon him. The quick, strong, successful pupil should have as a reward for good work not praise, but quiet congratulation and the opportunity of acquiring more knowledge. Perhaps the teacher will say to him something like, "You got it, didn't you? Tomorrow you can try this piece of work. It is still harder." Slow, weak, unsuccessful pupils should have not blame, but sympathy, expressed more in manner than in words, recognition of their small successes, and also sufficient help. They need frequent opportunities for steadying their minds against the stronger, clearer mind of the teacher. They should have direct, personal instruction, lasting only a few min-

utes, given when the need appears. The plan of giving your successful workers special work while you help the unsuccessful, suggested on a previous page, cares for both kinds of workers. Another way of caring for both kinds is to set them to working together. For instance, if two pupils, one strong in his work and the other weak, build the table of tens on the tagboard, working together, each is helped by the other. Especially is the weak pupil helped by the strong. Of course you understand that the building of the ten table is not an end in itself, that it is merely a means of putting the children in contact with certain numbers in a way that brings into their minds many vivid perceptions of the relations of those numbers. Care must be taken to see that the strong pupils are fair in allowing the others to have their full share in the occupation. In this way of working there are many beautiful chances of bringing out kindly feelings towards the child who has missed work by absence or is in delicate health or is in special need of kindness for any reason. Of course children are not angels and sometimes complications arise from the clashings of the wills of the little people. In times of peaceful activities the ideals of harmonious social work and play are held up, but when conflicts arise, the teacher promptly takes practical measures for securing harmony by separating or perhaps segregating, for a time, those who "don't play the game" with due consideration for others.

Educators generally agree that formal arithmetic is not suitable for children of the first and second grades. For this reason it has been taken out of those grades in many of the best schools. But number games and occupations, skillfully guided, are not only pleasing to little children, but *rightly used,* they lead to a realization of numbers that cannot possibly be gained by the old-fashioned routine drill and enforced tasks. Each game and occupation has a specific purpose, is intended to lead the pupils to grasp some particular idea. The skillful teacher knows definitely what result she wishes to obtain in giving it, just as the skillful physician, administering a drug, knows the result he desires to obtain by its use. All the games and occupations are carried on in the most leisurely way, not only to prevent errors in the work but to prevent the over stimulation of the children's minds and nervous systems. Awful things used to be done to the nervous systems of children in schoolrooms before practical school work was influenced by the kindergarten movement and the child study movement.

Queries: Who originated the kindergarten? Who is the leader of the child study movement in America?

Every day as the children come back to their happy social play with the numbers they gain a clearer idea of them in their unchanging sequence. Gradually the mental picture of the series is formed and becomes a part of their mental furnishings. They can call it up with more or less distinctness just as they call up mental pictures of their homes or school. A child of six or seven who has lived for sometime in a large, amply furnished home knows the forms, the locations and the names of hundreds of objects in the house. When sent to get a piece of music lying on the piano he does not make the mistake of going into the kitchen after it, because his mental picture of the interior of the house locates the piano in a certain part of the parlor. No one has drilled him upon the names, places and forms of the articles in the house. He has learned the forms and places, without conscious effort, by contact with the objects often repeated. He has learned their names by auditory contact, by hearing them often applied to the objects. And so in our early games and occupations we put the child in contact with numbers, in an agreeable way, in order that he may learn their names, their places in the series, and the forms of their symbols. While gaining this knowledge he is unconsciously learning many number facts, just as by his ordinary play he learns without effort many important facts about this great complex world in which he is an active and inquisitive newcomer. Conscious work and drill come later. Just now, you know, I am writing to you about beginners making their visualizations of the number series by means of plays and occupations.

Learning To Apply Number

Let us go back to the first lesson. In connection with counting by sound and with chart work the young teacher is also leading her pupils to realize number by counting objects. The children have been in contact with objects all their short lives, but the idea of number in connection with them, except "two" or perhaps "three," does not come clearly to a child during the first few years. Then the number sense awakens and gives him the new enjoyment of putting "the invisible band of number" around things.

Miss Smith's pupils count shoepegs, button-molds, pencils, coins, inch-squares and triangles made by cutting inch-squares diagonally. They count circles and half-circles. These geometric forms are arranged in patterns. Sometimes the teacher suggests the patterns, sometimes the children are allowed to use their own taste. The counting idea is always made prominent. Slips of paper upon which large dots are arranged in forms like those on dominoes are given to them and they report the number of dots. (See page 12 of state textbook for illustration.)

Groups of lines, horizontal, vertical or slanting are drawn on the board by the teacher or by a pupil and the children tell how many lines in each group. They count the desks or tables in their schoolroom. They find the number of pupils in the class. Some day the teacher says, "Tomorrow I want you to tell me just how many steps you walked upon when you came into the school building."

At another time they report the number of windows in a given side of a particular building on the campus. Of course, not all of the children remember to count them, but with the consent of the grade supervisor the class makes a pleasant little excursion outside and every child verifies the count of those who did remember.

It is not suggested to the pupils to count their fingers, and if they begin to do so, Miss Smith quietly diverts their attention. She does not wish them to form the habit of finger counting.

Not only things seen are counted but things heard and felt. The teacher taps on the desk with a pencil and asks pupils to tell how many taps were given. Turning to a pupil, she says, "Frank, you tap. Not more than five times." He gives four taps. "Now two more," says the teacher. The class report.

At another time she says, "I am going to clap my hands. You tell me how many times I clap them. Everybody look away from me." After the pupils have responded, she calls on members of the class to clap their hands and allows the others to report the number of times.

She sets them to marching around the room, saying, "Now you march until I say 'Halt,' and then you must be able to tell how many steps you have taken."

She says to the class, "I want some one to come out here and shut his eyes and see if he can tell how many shoepegs I put into his hand." So uncertain are the reports of the untrained tactile sense that this amounts to very little more than a guessing game. To prevent the class from getting confused ideas, she is careful to have the child whisper to her the number of pegs that he thinks he has in his hand. She writes the number and when the pegs are shown and the real number found, the teacher merely says "Right," or "Wrong."

Your own invention will furnish you plays enough when you get into the spirit of the thing. The only requirements for a good little number play are that the children like it and that it teaches them something in the right way. Plays in which the attention is not focused upon number, as in some boisterous number games, are objectionable. "Choosing sides" is not to be recommended because of the rivalries and jealousies it induces, and because as a result of these rivalries the zest of the play depends upon the making of mistakes by the opposing side. The children listen eagerly for those mistakes and naturally all except the clearest minded pupils get out of the play an uncertain mixture of errors and corrections.

In the exercise just described a group of things was to be observed by the pupils and they were required to tell "How many?" The reverse activity of giving to the pupils a number and requiring them to select the corresponding number of things is used in many ways. For instance the teacher or a pupil names a number and the class arrange on their desks the indicated number of squares or circles or other objects.

The pupils "tap out" or "clap out" numbers which the teacher or a pupil has named.

Sometimes they play a "mum game" with numbers. Instead of naming a number the teacher or a pupil writes it on the board or holds up a card upon which it is pasted. The pupils tap it out; not a word is spoken. A mum game, played not often and for only a short time, makes for quiet and self control, but it lacks vitality. When Miss Smith sees that the value of each number in the first decade is well sensed by the children, the plays are dropped and the objects are put away.

She uses a counting game a little more advanced called, "Flash." It is played with squares (or circles) of which some are of a dark color and the others light. This game is useful only for small groups. Miss Smith calls up three or four children who stand around her as she sits. The remainder of the class are at the board writing numbers. Their turn comes later. She has a boxlid into which she puts, perhaps, three dark squares and two light ones. The children, with the exception of John, whose back is turned because he is to be the first to play flash, watch her and she consults them as to how many squares shall go into the boxlid. When all is ready she says, "Flash, John." He turns, looks quickly and reports, "Three dark and two light. Five."

It will be seen that here is a little anticipatory work for the learning of combinations, the securing of a passing perception of the fact that three and two are five. Drill upon such facts and enforced tasks upon them at this stage of the learning process would be very harmful. The children need to stay in the concrete some time longer. Probably some child will say with the air of a discoverer, "Yes, three and two are always five." The wise teacher will smile encouragingly at this bit of generalization, the forerunner of many others, but she will not yield to the temptation to drill the class upon it. There must be many such spontaneous observations on the part of the children, much storing of their subconsciousness with number facts, many resulting happy reports of their little insights into number before they are ready for formal drill. Then they will welcome it as a help in fixing in their memories facts of which they feel themselves to be the discoverers, but which, very much to their regret, they often lose. The enforcement of tasks will not be necessary. Voluntary effort will take the place of it in the case of children capable of learning.

Does all this seem to you impracticable, rather soft-hearted idealism? It is not. On the contrary it is a bit of the most practical, hard-headed sort of thinking. It is not sentiment nor visionary idealism that prevents Luther Burbank from tearing open rosebuds in order to get fine, full-blown roses. His practical knowledge of the laws that govern those living organisms called plants leads him instead to wait for their development, in the meanwhile protecting them from force and keeping them in the most favorable conditions of soil and sun. And it is practical knowledge of the laws governing the minds of children that leads a teacher to wait for the natural, unforced development of the mathematical sense of her pupils, keeping them in the meanwhile in the most educative environment possible and in the sunshine of her loving expectancy. This is not a new theory. It is centuries old. In the past its application has been hindered by many conditions now disappearing under the new, strong demand for efficiency in school work. In differing forms but with the same spirit the theory is applied also in other lines of study here in the Training School. You would find it the country over, wherever the schools are controlled by advanced educational thinking. Many communities are "not yet ready" for that kind of control of their schools, but it is only a question of time when they will be.

Running thru all the work here described, recurring again and again, are three ideas: Visualization, Voluntary Effort, and the Storing of the Subconscious Mind.

The visualization of the number series secured by chart work and apparatus work is useful merely as a stepping-stone to an accurate and ready knowledge of number facts. This is the ideal towards which we are leading our pupils,—a knowledge of number so clear and strong that the mind seems to respond automatically to any demands for number facts such as are used in ordinary calculation. For instance when the question comes, "How much is 7 times 8?" the well-trained mind responds instantly, automatically as it were, with the correct answer. It has become a matter of the reflexes, like walking.

In the learning of number facts there are two periods: first, the period of unsconcious learning by means of play and work with concrete material; second, the period of conscious, voluntary work and play directed to the memorizing of the required facts. There are forty-five combinations to be learned, such as "4 and 3 are ?," with the corresponding separations, as "4 and ? are 7." In other

words, addition and subtraction. There are eleven multiplication tables each with its eleven facts, and there are the correlative facts of the division tables. It is no small task for children with their immature minds to perceive and fix in memory these three hundred and more number facts. It has been called an "insufferably tedious task." Fear, hatred, and dread of the work have been supposed to be the necessary accompaniments of the drudgery it involved. But now just as you fortunate young people are coming into the profession, plans for adapting the work to the capacity of the learner and for utilizing the play spirit are becoming general. As a result the fear, the hatred, and the dread are disappearing and interest and pleasure appear. Some children become too much interested in number work, just as some people become too much interested in solitaire, which is after all merely a grown-up number game. Such children should not be praised and pushed forward. After they have done a fair amount of daily work in number their attention should be diverted to other interests.

In the beginning of the first period, the purpose of the teacher is to arouse in her pupils a sense of the relations of numbers by means of play and work with material, with charts, and with different kinds of apparatus. Most of the apparatus used in our classes indicates number relations in the same way as the chart, but in a more objective form. During this time children often show that ideas of the relations of numbers are forming in their minds, by their reports of their little discoveries made while using a chart or some piece of apparatus or while "thinking about it at home." A child not yet six years old who had played with a chart a few times called his aunt's attention to the second horizontal line of the chart and remarked, "All the numbers that have 2 at the end live on the same street." Little ones sometimes show reflective thought by such questions as "Where do the numbers go after they get to 100?" or "What is the very biggest number in the whole world?" By means of the play and the work and the quiet. happy thinking during this period the children are led to perceive the facts of number as a series of related facts. This effects a great saving of time and effort in the later learning process, as compared with the plan of learning them as independent, arbitrary statements. If in studying geography a child should be set to learning a great many separate. unrelated facts, such as, "Chicago is west of New York," "Denver is east of San Francisco," etc., without any map to show directions and relative distances. if he simply memorized them as independent, arbitrary statements, he might be drilled upon them for a very long time without getting a clear idea of the situation of the places mentioned in these assertions. The facts would be likely to slip out of his memory in the way expressed by one of our student teachers last year when she said, "What I teach these children in the day they forget in the night." Now the magnitude of a number is known by its place in the number series. For an illustration let us think of the numbers 19 and 64. From the fact that 19 comes earlier in the series than 64 we derive the idea that it is less than 64. A child playing with charts, number squares, etc., soon gets similar ideas, because on the chart the relations of numbers are shown by the directions and distances of the printed numbers from one another. The chart gives the same kind of help in the learning of number facts as that which is given by a map in the learning of geography. For instance, a child in the playing stage of number learning sees that 6, coming farther on in the series than 4, is a bigger number, means more things than 4. As he becomes more definite in his thinking he sees that 6 is just two steps beyond 4 or that "4 and two more are 6," and that "4 and three more are 7." Soon he sees that 26 is just two steps beyond 24 or that "24 and two more are 26," and so on. Then he is thinking number intelligently. He is getting ideas of related facts to be used with clear perception and strong memorization in his later work. Hence the importance of this first period of play and occupation. It is also important that this period should come in the early grades while the child is forming his concepts of number, at the time when his interest in number and his desire for childish play are naturally strong.

In the latter part of this period the pupils begin to acquire the power of visualizing the number table on the chart, of intentionally "thinking how it looks." At this time there is much quiet, ruminative thinking on the part of the successful pupils. They have the visualized series at their command and they make many observations upon it which they report in class. After the first decade is mastered some one is almost sure to say, indicating the 10, 20, 30, etc., of the chart, "See, the 1, 2, 3, 4, run right along on the bottom line, too." As a little fellow in one of last year's classes remarked, while working on the decades, "We are using the same old figures right over again." When the eleven chart, given below, is first presented, usually some pupil exclaims, "The elevens all run down hill."

1	**11**	21	31	41	51	61	71	81	91
2	12	**22**	32	42	52	62	72	82	92
3	13	23	**33**	43	53	63	73	83	93
4	14	24	34	**44**	54	64	74	84	94
5	15	25	35	45	**55**	65	75	85	95
6	16	26	36	46	56	**66**	76	86	96
7	17	27	37	47	57	67	**77**	87	97
8	18	28	38	48	58	68	78	**88**	98
9	19	29	39	49	59	69	79	89	**99**
10	20	30	40	50	60	70	80	90	100

When the nine chart appears they are ready to see that the multiples of nine "run up hill." The spontaneous, undemanded expressions of the pupils in regard to their work are pretty good indications of the vitality and success of your own efforts in teaching.

As you have been in Prof. Baker's classes and have studied his manual, you know the value he places upon correct mental picturing at this stage of the work. The usefulness of these mental pictures is also clearly recognized in a recent book, The Teaching of Arithmetic, by Dr. A. W. Stamper of the Chico Normal School.

In this visualizing work it is the aim of the teacher to make her pupils absolutely independent of all concrete aids to thinking. Gradually she leads them from the use of the chart to the use of mental imagery. "I wish I had a chart like this, at home. Then I could reckon anything I wanted to," said a boy in the second grade, beginning number. "You don't need it," replied the teacher. "You will soon have one in your head." Her prophecy came true. In a few months the boy was able, while standing with his back to the chart, to answer such questions as "How many are 96 and 3?" "50 and 20?" He answered these as correctly, tho not as quickly, as he would have answered the question "How many windows are there in your mother's kitchen?" and by the same process, that of forming a mental picture.

I hope it is clear to you that this visualizing stage in which the learner brings up a mental picture of the number series and upon it counts out facts just as he formerly counted them out on the chart, is only a passing phase of the learning process. It is intermediate between the counting on the chart and the quick,

sure knowing of the required number facts. After that is reached there is no need of visualizations of the number series. If any exist they lie unused in the background of consciousness, just as the mental picture of the typewriter keyboard, so necessary at first, is unused by the typist after she becomes expert. But while the child is in the visualizing stage we must see to it that his mental pictures are correct and vivid. There are several little exercises designed to secure good visualizations, which I will show you in your classrooms. None of them, however, are as effective as the simple device of keeping a plainly written number chart always on the board in convenient range of the pupils' sight. The silent, unremitting instruction, sent by the chart into the minds of the pupils whenever they chance to turn their eyes towards it, greatly shortens the time and effort necessary for the conscious learning of number. As Prof. Baker expresses it, the children are "exposed" to mathematical truth. Much is absorbed by their impressionable minds, and it is stored in the subconsciouness.

The second period of number learning, that in which the child is consciously acquiring the knowledge of number facts that is to serve him the rest of his life, lasts two or three years. At this time the purpose of the teacher is to help her pupils to become so prompt and accurate in their responses to number questions that their mental action seems to be automatic. The time in which this power of reflex response to demand for number facts can be acquired differs greatly in individuals. It cannot be shortened by outside force as that only produces bewilderment. Hence the need of waiting for the individual ability to develop. Some children, not well equipped by nature, can not become quick and accurate in number work except at an expense of time and effort greatly disproportionate to its value. For them the "minimum essentials" mastered and managed in their own slow way are all that are feasible. The power to reckon quickly and accurately is soon lost by disuse just as facility in piano-playing or in typewriting is soon lost when practice stops. Of course it may be regained by renewed practice. This power of ready response to number questions can never be gained by a learner whose perceptions of number facts are vague or incorrect. Hence the importance of vivid and accurate presentations of those facts.

There is a theory well known to educators, the theory of "brain paths," given by Prof. William James in his great work, Principles of Psychology. He presented it as a hypothesis used to coordinate many known facts about the way in which people learn. Psychologists agree that all of our thinking is accompanied by motion of the molecules of certain cells in our brains, and that this motion passes along from one cell to another. Hence the name "path." In the light of this theory we shall see that it is our bounden duty to insist, from the beginning and all thru the work, upon absolute correctness, secured by slow, clear perceptions and by careful expression. According to this theory the first time a child thinks, "5 and 3 are 8," he starts a brain path. Every repetition of that fact deepens and smooths the path, as it were, and makes the recalling of the fact more easy and swift, until after many repetitions the action of the brain centers involved becomes reflex. But if he makes the statement that 5 and 3 are something else than 8, or hears it made by his mates, a new, diverging brain path is made, along which impulses are likely to travel whenever the question is asked, "How many are 5 and 3?" Uncertainty begins, uncertainty that sometimes lasts until the pupil has reached high school work, as many, many teachers besides myself can sadly testify. To prevent this uncertainty, or any kind of mistaken thinking about number facts, it will be necessary for you to insist that every child, when in doubt, shall take the truth at once from the visible number series of the chart, without giving time for any wrong impulses to play thru his brain. We say to our pupils. "If you think mistakes when you are little it will hurt your brains and make you stupid in arithmetic when you get older, and if you say mistakes out loud you will hurt the other children's brains." This is exactly true, and we find it very effective in arousing in the pupils' minds a strong

aversion to mistakes. In this way the "storing of the subconsciousness" with errors can be prevented to a certain extent. Of course it will be impossible for you or anyone else to prevent it entirely.

When the work of memorization goes on successfully the interest heightens. The joy of acquisition is aroused in the children. They rejoice in the possession of their bits of number knowledge as the miser counting his coins, rejoices in his possessions. They express their feelings in such remarks as, "I know the even numbers and I know the table of tens and I know a lot of the fives." Sometimes they ask for drill for some particular purpose. For instance, a child who wants to get more fives or to keep what she has got will perhaps ask the teacher for a class drill upon the table of fives. It is a useful plan to have the pupils make in class little books in which they write the number facts that they are sure they know. These little books, made by folding a few sheets of paper, and having a title like, "My First Number Book," should of course be as neatly made as possible. When filled they may have a cover decorated according to the fancy of the owner or the taste of the teacher. They should be exactly true in their contents. For this reason the pupils should put into the books only a very few facts at a time and those should be facts upon which they have "stood test" several times. Of course the teacher will carefully examine the books to see that no errors creep in. If any are found the teacher should erase them without mentioning them. In the next class period she will question the pupil about the fact, and when she is satisfied that he has it clearly in mind she will allow him to put the statement of it in the vacant space.

My dear students, in this letter many topics are touched upon, ranging from playthings to psychological theories, and I know that many of the ideas will not be clear to you until they have been elaborated in your professor's classroom and in our weekly conferences or until they have been worked out in your own classrooms. If you will read again, and as a continuous story, the descriptive work printed in small type, you will see that it merely covers, in sketchy outline, the learning of the three first essentials mentioned on page 3, the sound series, the sight series, and their use in applying number to objects. Next comes the understanding of the decimal notation with its recurrences and repetitions, and along with it the writing of numbers beyond 100. It is not worth while to present any more descriptions in this letter to you, because a few minutes of demonstration work, such as I hope to give in your classrooms from time to time, will help you more than dozens of pages of description of it, besides being much easier for you, and for me. The book which I am writing about the happy learning of mathematics, a copy of which I hope to place in our library in the course of a year or two, contains a great deal of description of schoolroom activities. It is not the purpose of this letter to give you a set of pedagogical devices. Its aim is to lead you to think intelligently about the principles, purposes, and reasons for procedure, that underlie the work. Plans, methods, devices, apparatus, all avail little or nothing unless the true spirit and understanding are in the teacher. When these come to you and with them the skill to adapt the work to your own classes you will find many plans useful to you in books, and especially in school journals.

The learning of the facts and processes of addition, subtraction, multiplication, and division is the principal work of the third and fourth grades. During this time the pupils are also forming clear, elementary ideas of multiples, factors, fractions, ratios, measurements, and geometric forms as presented in the drillbook used in the third grade. They are also learning something about the applications of numbers in daily life. As soon as they have a few number facts they begin to use them in number stories. (See page 29 of state textbook for illustration.) They play store, count money, and make change. They report actual purchases that they have made. Suppose Hugh has bought a pencil, costing 5 cents. The class may tell the different coins used in making change if a quarter was offered in payment or a half dollar, or a dollar. Care must be taken

that the children do not make disclosures of family affairs in their eagerness to report actual buyings in which they are interested. Other applications of number may be drawn from their work in manual training, in construction, and in other departments, but the main stress should be laid upon the direct memorizing of facts. This is to be enlivened by many number games which will be shown to you.

No formal analysis, no problems requiring the consideration of "steps," should be given at this time. This is the time when the perceptive and retentive powers are to be used, not the powers of formal, abstract reasoning. In the training of children for circus performers the exercises are carefully adapted to their powers. Apart from considerations of humanity the trainers are unwilling that a child should be overtrained or made fearful, as that would probably prevent him from becoming a fine adult performer. Long ago Pres. Eliot pointed out the fact that if children are to become clear-thinking mathematicians in their later years of learning, they must not be given, in their early years, exercises that involve bewilderment and struggle. Many of the failures in the higher grades are due to this cause. Many pupils enter those grades who, instead of having a clear knowledge of number which they can use in connection with the reasoning processes, have an abounding hatred of it as the chief result of their early unsuccessful struggles with the subject.

In the pre-memorizing period of working with the combinations the child has them all before him on the chart, and he uses them in various ways; as in the exercise of "telling combinations", in which each child has his turn in selecting combinations from the chart or, if he is very sure of himself, from his mind. But when the work of memorizing begins, the combinations are not to be presented in a haphazard way. Instead, a few are selected and they are thoroughly learned and presented as a stunt by each child. Games, drills, tests, boardwork, bookwork, all bear upon the set of combinations which is being considered. In this way the desire of the pupils to acquire it is aroused. *And their desire is the great impelling force in all this work.*

The first set of combinations consists of those of the even numbers. It is preceded by counting by twos, and by games with even numbers. Let me caution you, if you want to avoid mix-ups, don't use the word "odd" in connection with the odd numbers at this time. The numbers are even or not even. Months later when the distinction is clear in the minds of the children, it will be safe to use the word "odd". When pupils are able to add 2 to any even number that they can see on the chart or can think of, they practice adding 10 to even numbers as well as 2. It is easy to add 10 on the chart, as it requires only a move to the right on the horizontal line, a fact which the pupils soon discover. 4 is taken next. They then have the three numbers, 2, 10, and 4, which they use as addends with other even numbers. They naturally want more. 6 is given to them and they are set to ringing the changes of these four numbers upon the even numbers. When they are expert in this, they take 8. This completes the first set of combinations. Then come the corresponding separations. The use of flash cards, of test cards, and much written work is needed. Column addition of even numbers and the addition of two numbers in the thousands, all of whose digits are even numbers, are used at this time. Keen-minded children soon discover a law which one expressed as follows: "If you add an even number to another even number you will get an even number, and if you keep on, you will always get even numbers. The other numbers are not so." A little girl remarked, "The other numbers are just put in between the even numbers to hold them together." Upon this a boy said that he thought "they were put there to keep the even numbers apart." The teacher could not decide, but she was greatly pleased with these comments because they showed that the attention of the children was focused upon the even numbers, which was just where she wanted it to be at that particular time. When the class has stopped blundering or doubting on this set of combinations and separations, it is ready for the second set. Probably some individuals, who got ready before the class in general, are already preparing stunts in advance.

The second set consists of the combinations that make the even numbers thru 18, group combinations, as 4 equals 2 and 2, and also 3 and 1, etc. I hope that those of you who have this work this year will remind me to give you the game "How many?"

The third set includes all the other combinations; that is, the groups that make the odd numbers thru 17.

It will take many months for the pupils to master the combinations and separations. If the teacher hurries and crowds the work, the pupils will become confused in their thinking and it will take many more months. It is a common mistake of inexperienced teachers to suppose that when a pupil has recited a fact he will continue to know it. The first learning is always temporary memorization, scarcely more than perception. There must be many repetitions of it at different times before the brain path is worn deep enough to be permanent.

I do not believe that you are able to tell what combination of foods composed your dinner a week ago last Thursday. You had a vivid knowledge of it at the time. All your senses reported it to you. (Observe that in learning number only visual, auditory, tactile, and motor impulses are involved.) But I venture to say that your knowledge, vivid as it was at the time, has proved to be only a "temporary memorization." If you were to have the same combination of foods at many meals, your memorization of them would be more lasting, its permanence depending upon the number of the repetitions.

Query: What does this illustration suggest to you?

During the months in which the facts of addition and subtraction are the central thought of the work, the facts and processes of multiplication and division are also being comprehended and to some extent memorized. The multiplication tables are taken in the following order: tens, twos, fives, elevens, nines, threes, eights, fours, sevens, sixes, twelves. The reasons for this order will appear in the work. They are too lengthy to be given here.

It is a fact well known to teachers of mathematics that pupils can carry on two or more lines of activity at the same time without becoming confused, provided that one does not depend upon another. For instance, if we were to introduce the second set of combinations before the first was mastered, confusion would result; but when we allow the pupils to work on a multiplication table, it does not interfere with the learning of the combinations and it is a pleasing bit of advance. A day's lesson, which lasts forty minutes in these grades, may include, besides combinations, work and play with a multiplication table and with geometric forms in which doubling, halving, and otherwise varying the forms is a feature. On another day simple ratios will be presented. The pupils can see from the chart that 10 is one half of 20, just as one column is one half of two equal columns. When the ideas of one third, one fourth, etc. are grasped in the work with geometric forms, the fact that 10 is one third of 30, one fourth of 40, and so on is equally clear. The day's lesson should usually include a little new work carefully adapted to the children's comprehension. As an opposite kind of activity, some snappy work for speed, judiciously used, is helpful and enjoyable. It must not be carried to the point of over excitement. The last twenty minutes of the period is devoted to written work from the book. Then is the teacher's opportunity to call to her some pupil who needs a few minutes of help. Perhaps there is a boy who is "a problem," indifferent, careless, out of harmony. As he stands by the teacher's knee, as he would by that of his mother or father, and receives the friendly help, he may become interested, perhaps softened by the mind-to-mind contact over the work. It is worth trying.

About plans. You know, of course, that a teacher who goes before her class without a definite purpose to teach some particular thing or without tentative plans for teaching it, is a failure from the start, just as your dressmaker or tailor would be if attempting to make a suit for you without having definite plans for it. Your daily plans should be made on the day before they are used. For instance, in your preparation period on Monday, while Monday's lesson and its immediate results

are still fresh in your mind, think over the plans for the next day's lesson, select the exercises that you propose to use and make a list of them. Have this list for the day on your table, where it may be consulted by the grade supervisor and myself when we enter your room. We trust that it will be read occasionally by the principal and by the head of the mathematics department, who are also interested in your work.

Instead of a written plan for the coming week please hand to me at the weekly meetings a written report of what you have done in the past week and the results of it as far as you are able to determine.

As soon as you have decided upon the seating of your group, please make a diagram of it, giving the full names of the pupils in the order in which they sit and your own name and grade. Hand this to me at your first opportunity.

In your few weeks of practice study in the Normal you are working under favorable conditions created especially to enable you to learn the ways in which children's minds react under teaching. Many of the responsibilities which will be yours in your future work in the schoolroom, here devolve upon your supervisors. In order that we may be helpful to you and may protect your pupils from any harm that might come to them from the unguided efforts of inexperienced teachers, it is necessary for us to know your work as closely as possible. As your success means the success of the children and also our success, many earnest and kind wishes are centered upon you.

With pleasant anticipations of our work together, I am

Your sincere friend and co-worker,

ADELIA R. HORNBROOK.

San Jose, Cal., Sept., 1913.

ERRATA FOR "OPEN LETTER"

Page 4, 47th line, 5th word, "deficiencies."

Page 10, 39th line, 8th word, "made."

Page 19, 1st line, "rest of the" should be between the 6th and the 7th word.

III.

THE FIRST FIVE YEARS OF ARITHMETIC

Second Letter From a Supervisor

MY DEAR STUDENT TEACHERS:

It is important that each of you shall understand the general course of instruction of which your term's work is a component part. This chapter contains a brief outline of the number play and work of the first two years and of the more formal work of the three succeeding years at the end of which the elementary state textbook is completed. It also contains many suggestions for work which heretofore have been given orally.

First Year Twenty minutes a day are given to number play consisting of occupations with simple apparatus, plays, and games. The general purposes of the first year activities are to lead the child to know the sequences and values of the first hundred numbers, to give him motive and opportunity to perceive and use many independent number facts, to help him to memorize the combinations whose sum is 10 or less, and to enable him to read and write numbers as far as thousands.

As the opportunity for voluntary effort is always open to the children some of them learn and present as "stunts" much more than others. At the end of each term the classes are sectioned so that the quick pupils are in one group and the slow are in another, no idea of inferiority or superiority being attached to either set of pupils. If you are put in charge of a section of slow pupils, you will have the opportunity to use much sympathy, watchfulness, and ingenuity. It is often the case that the "slow" pupil is merely the unawakened. If the awakening of the mathematical sense comes while he is under your care, you will have the pleasure of assisting in rapid and happy development.

In the "Open Letter" which forms the preceding chapter of this manual some beginnings of the plays, occupations and games of the first year are described. (See pp. 3, 4, 5, 6, 7, 8, 9, 12, 13.) These descriptions are given as a basis for your work. Follow them closely at first until your teaching sense begins to develop. Then vary the plans to suit the occasions that arise.

Each exercise has its specific purpose. For instance, such plays as "hunting", "hiding", and "letting out" numbers, described on pages 7 and 9, are designed to teach the child how to read numbers. They also help him to realize the value of numbers by fixing in his mind their places in the series, as shown in the chart.

For the beginning work desk charts are to be prepared, one for each child. (See description p. 8.) In the 1B grade, only the first decade should be written in the desk charts. The second is added after the first is learned, and so on. Keep a large chart of fives always upon the board. The other apparatus must be prepared when needed, except tagboard, buttonboard, nailboard, decade sticks, abacus, etc., which are furnished by the school.

In all grades each teacher before beginning work should place her plan for the day's lesson upon the supervisor's desk and should see that all the material required in it is at hand. It is as absurd for a teacher to go before her class with only her voice for a tool as it would be for a carpenter to attempt to work with only his fingers, having neglected to bring his tools. Among the tools of the teachers of the first and second grades are charts, number squares, crayons, eras-

ers, pencils, paper, large cards with figures for tracing, small cards with figures for games, cards with domino spots, flash cards, buttonstrings, dissected charts, rulers, rolls of ribbon paper, objects for counting and combining, as squares, circles, semicircles, triangles, star points, leaves of trees (especially the peppertree), toothpicks, buttonmolds, shoenails, etc. Charts, crayon and erasers are *always* needed. From the others the teacher should choose a sufficient number of those needed to carry out the plan of the day's work.

Fresh air is so important in the schoolroom and our soft climate allows us to obtain it so easily that your first care upon entering your classroom, whatever may be your grade, should be to see that your windows are rightly adjusted.

Definite plans and an abundance of material, altho they are essential, are not enough to insure success. I have seen in the classroom most dreary, perfunctory, futile exercises which were sincerely intended for number play by the mistaken young teacher who carried them on. She had her plan and her material but at least two great essentials for success were lacking in her mind. The first was the play spirit. Without the spirit of play all this number play is worse than useless. "How can I get the play spirit?" do you ask? It can be acquired in various ways. Watch children at play and try to interpret their actions, play with them, recall your own childish plays and feelings, talk with those who play with children successfully, as kindergartners and playground supervisors. Read books and articles upon the subject. For your immediate purposes when you are appointed to a number play section, if you take into your classroom the plays suggested in this manual and shown in the supervisor's meetings and enter into them wholeheartedly, watching the children, thinking about them, trying to interpret their feelings, you will find the spirit of play coming to you. It is natural for children to love to play. And it is just as natural for women and for many men to love to play with children.

There must also be in the mind of the teacher clear ideas of numbers in their relations to one another and a strong desire to lead her pupils to realize certain bits of number knowledge by means of the happy, purposeful play. This desire will lead her to watch the actions of her pupils' minds and to devise means for reaching them. She will strive to adjust the play to the needs of every little member of her flock.

Every term I watch with delight the development of teaching power in students who, after a period of apparent daze and weak, uncertain effort, catch the spirit of the play, and as they grasp its underlying purposes become so filled with desire, determination and expectancy for their pupils' happy learning that not only is the success of the pupils assured but their own. As I write, a mental picture of a scene in the classroom of one of these successful young teachers comes before me. She was a quiet, soft-voiced woman. There was no affected sweetness nor forced jollity in her manner. She was deeply interested in her work and acted naturally. In her directions to her class she was clear, confident and exact. She did not harangue her children, nor shout orders at them, nor exhort them to "be good" or "pay attention." As I entered the room one day she was sitting at one side of the large table with the children. They were playing with the abacus. (Ordinarily an abacus has twelve rows of beads, but ours have been made to correspond with the chart by clipping the beads from two of the wires.) She was carrying the children's attention in an easy way. She was not discouraged by the lapses of the little brains. She merely led the wandering attention back to the desired point by directly and pleasantly including the inattentive little one in the play. In the course of it a boy, whom I will call John, announced, "When you have ten beads on the abacus and take ten more you have twenty, just like the chart. When you have ten numbers on the chart and then ten more you have twenty." He made his meaning clear by showing the first two decades on the chart and pointing out the 20. "Did all of you hear what John said?" asked the teacher. "Tell it again, John." John told it again more definitely and with more delight than before.

The teacher had planned other things for that lesson, but like the wise little woman that she was, she seized the opportunity thus given. She led the class into a discussion of "what John found out." They showed on their desk charts the fact that 2 tens make 20. John proudly handled the abacus, that privilege being the natural reward of the discoverer. It was developed also that 3 tens are 30, and that 4 tens are 40, ("Don't you see the 4 in the 40?" was asked,) and so on. Not all the children grasped the idea that day but it got into their mental atmosphere. Sooner or later as the subject came up they all realized the facts and hence remembered them in a way impossible to them if they had been set to memorizing the table as a mere patter of words.

When a child sees for himself that the 3 in 30 means 3 tens, that 6 tens are 6-ty, 7 tens are 7-ty, etc., then (and not until then) is he ready to drop the use of the concrete and to take up with ease the conscious learning of the multiplication table of tens. "Standing the test" upon this table and later upon other tables are achievements in which children take much rightful pride.

Note carefully that what is meant by the "table of tens", as presented to beginners, is the series of statements, "1 ten is 10," "2 tens are 20," etc. To the children each ten is an entity, a ten of beads on the abacus or a ten of circles on their table or of numbers on the chart. A statement about 5 tens has for them as concrete a basis as a statement about 5 sticks. The reversed ten table, "10 times 1 are 10," "10 times 2 are 20," etc., does not deal with tens. It deals with ones, twos, threes, etc. This reversed ten table is of course necessary to be known, and we lead up to it by a gradual process of reversing the known facts of one table so as to correlate it with the others, or as we call it in the classes, by "finding old friends" in a new table. In all grades where the children are learning the tables of multiplication and of division, teachers should be sure that they are using the table for which the children's concrete work has prepared them; otherwise the repetition of the statements becomes mere word patter.

I know that you are handicapped in the number play by the fear that the discipline of your room will not be satisfactory. But your department supervisors are eager to have the children led into knowledge of number by organized, well-directed play. They comment happily and enthusiastically upon the success of those student teachers that accomplish it. It is necessary to distinguish between the informal play by which children learn number, and the formal discipline by which they enter and leave the room, or execute any concerted movements. This formal discipline is not only a necessity in the managing of classes of children but, rightly used, it is a great benefit to the children individually in many ways. One has only to look at a West Point graduate to see some of the physical benefits of discipline. The personal and social ideals inculcated in quiet, prompt, orderly, concerted action in the schoolroom are of great value. You can get this discipline by first finding out just what it ought to be and then insisting upon it from the first in a kind, absolutely decided way, taking if for granted that you have the sweet obedience and coöperation of your pupils in "doing things right." You must be able to pass at will from formal discipline to informal play and back again. Without this foundation of discipline, the details of which you must get as soon as possible from the department supervisors, the play becomes wild and purposeless, very tiring to the children (and teacher), and doing more harm than good to their number sense. Observe and use from the first the signal by which a child signifies his readiness to speak, the "position". This is for several reasons a great improvement upon the old plan of raising hands.

All the plays are very simple and are based upon well-known psychological principles which it is not necessary to discuss here. They make varied appeals to the senses and to the capacities of the child. He not only hears, he sees, feels, handles, constructs, chooses, counts, measures, draws, matches and ad-

justs, observes, discovers and reports, gives "stunts", keeps scores and reports them, tells stories and comments upon those of his mates, dramatizes, in short, uses his mind, his body and the number series in many interesting ways. And thru it all he has the joy of acquisition, of feeling that he is getting knowledge of number,—that wonderful thing that grown-ups seem to know so much about and that will help him to be like them when he learns it. None of the plays are boisterous altho there is much movement, sometimes rhythmic, as in pointing and counting, or marching and singing, and sometimes free, as when the children walk over to the corners of the table and count out for themselves the number of circles or triangles or other pieces needed for the patterns which they are making. This is much better than having them sit passively while you pass out the pieces, *provided* that the children are quiet and polite. It is well to tell them that this nice way of getting the pieces can only be used in rooms where the children are polite. It is well to mention the same thing casually to some grown person (as for instance a supervisor) in the presence of the children.

Absolute accuracy is insisted upon as an essential part of the plays. (See the story of Tom, p. 5, also theory of "brain paths", p. 16.) It is vastly better for children to be playing outside than to be in the classroom learning false statements about numbers. And if they are allowed to make errors and to hear them they will learn untrue statements. Of course you will not repeat an error nor call attention to it. Quickly and emphatically substitute the fact.

Among plays suitable for first grade pupils are Matching Numbers on the chart; Having Numbers Dance; Locating Numbers; Counting and Pointing on the chart by tens, or by fives or twos; Building Tens on the table with number squares, or blank squares, circles, semicircles or triangles; Building Tens with tags on the decade sticks, or on the tagboard, or with buttons on the button-board, with beads on the abacus, or with nails on the nailboard. After all these building occupations the game of "Telling Which" is useful, because it causes definite perceptions of the places occupied by different numbers in the series and leads to clear visualizations. When the desk charts become slightly worn they are replaced by new ones and the old charts are cut into strips each showing a decade. Putting these decades together in their original form is one way of Building a Hundred. Again the old charts are cut into strips each showing five numbers. These strips are used to build a hundred. Or the charts are cut into twos with which to build twenty or thirty or fifty, as the children may be able in the given time. Some multiple of 10 is always chosen as the limit of the building. After a time the system of decimal notation seems to dawn upon the children. Then they are ready for new ideas and new plays to lead up to them.

Running to a Number, begun in the first grade, is used with adaptations in the first five grades. This begins with "creeping." For instance, the teacher directs the children to put their left forefingers upon 3 on their desk charts. "Count on four," she says. With the left finger still on 3 they count and point out with the right the next four numbers and report "7." They hold the right finger upon the 7 until another addend is given. Then the left finger is brought forward in the counting and pointing and is placed upon the number reported. And so on until a desired number is reached. The teacher uses "Count on" and "Add" interchangeably until the former is dropped. Soon the children are able to count without pointing out each number. Then the small fingers no longer "creep" but "walk" over the chart. Later when the children have learned the distances and directions of the respective numbers the fingers "run", occasionally "jumping", as when ten or some multiple of ten is added. The children take turns acting as leaders in giving addends. But first the teacher leads, usually carrying some one idea thru the exercise. Perhaps she begins with 6, adds 4, then 6, then 4 and so on until some child sees and reports, "I don't have to count. When you add 4 to a number that has 6 at the end it always brings you down to the end of the decade where there is a 0." Of course he is invited to show the "easy way" to his classmates. Some of them see it at once. Perhaps some slow-

minded little one will report the same thing some time afterward with great delight, believing it to be his own discovery. Such a belated perception is warmly welcomed by the teacher when it does come.

To give a foretaste of subtraction the teacher directs the children to "run backward" on the chart. Then she sends them forward again, giving such directions as "Put finger on 5. Tell how many you must add to reach 8."

Later the pupils "run to numbers" on the wall chart. In the succeeding grades, after the mental picture of the number series is distinct (See p. 15), they run "in their minds". Still later when they reach the stage of automatic knowing, towards which all this childish but clear and happy perception work is designed to lead (See p. 13), the exercise has become for them really accurate and rapid addition and subtraction. Other ways of anticipating addition and subtraction are by means of various plays with tagboard, buttonboard, etc.

To make the pupils familiar with the combinations that make 10 is the purpose of several of the early plays. The "parting game", plays with the button-string, "fishing for tens", and the game "How many?" are among those used to give practical ideas of group combinations, not only of 10. but later of other numbers. The footrule is also used in finding group combinations.

When your pupils play these games quickly and surely, prepare a set of testing cards and call for volunteers to "stand test" upon the combinations that make 10. Then the separations. Keep records. Later let them try the combinations of other numbers. Remember that in every test "the first mistake spoils the stunt".

In "Making Patterns" definite numbers of geometric forms, as circles, half-circles, squares, triangles, rhombuses, etc., are combined into regular figures either copied or original, the number idea being made prominent.

The time during which the digits are presented (See pp. 7 and 8), should be at least three months. As the children learn to make the figures correctly, the making of charts upon the board or upon squared paper becomes a useful and pleasing occupation. After the pupils know the first hundred numbers and you think that they are ready for larger ones, if you will place a tag showing 25 over the two naughts on the 100 tag, usually some one will be able to read the 125 for you. If there is little or no response, wait a few weeks before trying it again. Do not let any one say "One hundred *and* twenty-five." Omit the "and."

The children play with even numbers in many ways. They march and count by twos, sing the even numbers, sort out even numbers written on cards, group objects in pairs and report the number of pairs, make two-charts with the first thirty numbers, play the game, "Hunting Even Numbers," in which two children go to corners of the room and hide their eyes while the rest select an even number for them to find from hints given them, or the game in which they "fish for even numbers." Running to numbers stepping only on the evens, is a useful exercise. Plays with even numbers have many variations but they all have the same purpose—to focus the child's thought upon the even numbers to the exclusion of the others during this time of preparation for the later work of learning the first set of combinations and the table of twos. Hence in playing with even numbers the odd numbers are never intentionally mentioned.

Gradually the children begin to remember some of the facts which they are perceiving and reporting in their plays. Then they are ready for the exercise, "Giving Number Facts". The phase of development in which children are eager to give out number facts which they have subconsciously stored, appears usually in the 1A or 2B grade. At first the teacher writes upon the board the facts given by the children, thus showing them the correct ways of writing their beloved facts. Soon, however, they write them upon the board, upon paper and in little blank books made by folding a sheet of foolscap. Filling a Number Fact Book is a fine occupation for those who are ready for the work. When the teacher sees an error on the board, she erases it and sets the pupil to verifying

his fact by "counting it out" on his chart. Papers containing errors are not returned to the pupils. When the teacher finds a mistake in a number fact book she erases it, or if it is written in ink she cuts it out.

Second Year Number play and number work go on in a leisurely way thru the second grade, twenty minutes a day being spent upon it. The play is lessened as its purposes are attained. The work is increased, but it is always adjusted to the stage of "readiness" of the class. The early occupations give way to such interesting exercises as working examples, applying numbers to geometric forms, discovering the fractional parts of surfaces, lines, solids, and numbers. The general purpose is to lead the pupils to memorize the first set of combinations and separations (those of the even numbers), and the tables of tens, fives, twos and elevens, and to learn to read and write numbers to millions.

There are usually some children in a class whose mental age is beyond that of their mates. A child of seven may have the mental power of an ordinary child of ten. Such children must not be held down to the work of their mates which they already know but must be allowed to learn and present as much as the time permits. Their work is not only a joy to themselves but it is an inspiration to their mates. In forming the habit of independent, advancing work they are laying the foundation of later success. While the teacher should not stimulate them she should give them from time to time the bits of information which they may need. This applies in all grades.

Giving number facts is an important exercise in the 2B grade. The children get their facts from the chart or from their mental pictures of it and from one another. Sometimes a child presents a particularly enormous fact about which he says " I got it from papa." (Of course you will look out for the weak ones and see that every one has some fact to present.) The little people like to deal with large numbers. Hence it becomes necessary to show them how to read and write thousands, and later millions. The Million Stick and the metal numbers are used for these purposes. Most pupils are eager to take up "easy ways" of getting number facts. For instance, they see that 10 can be added to any number without counting, by "just going straight across to the next decade". The teacher shows them such facts as that in adding 7 to 40, instead of finding 40 on the chart and counting on seven numbers to 47, as they have been accustomed, they can just put a 7 in the place of the 0 in the 40. Quick pupils see at once this short way of adding numbers less than ten to the multiples of ten, and the subject is brought up from time to time until all grasp the idea. Doubling Numbers and its complement, Halving Even Numbers, furnish many facts. When a child knows from his own experience that 5 and 5 are 10, it is not difficult for him to see that 5 is one half of 10. If you have the children place on their desks an odd number of circles, as five, and tell them to find half of the circles it will be interesting to see how many of them grasp the idea of halving a number that is not even. Be sure to put the results of such trials into your weekly report.

Column addition, the addition of numbers in the thousands, working without the chart, and other "grown-up" ways of dealing with numbers are very pleasing to the little learners at this stage. Addition with "carrying" is so desirable an accomplishment that sometimes they get it from their elders at home. Before they have learned carrying, it is necessary to use for addends only those numbers in which the sum in each order except the highest is less than ten, as 826 with 843. It is a good plan to call upon some child to give the first addend, supplying the other addend yourself. But after they have learned to carry, as they can get all needed facts from their charts, it is well to let them furnish both addends except when you have some special point which you wish to bring out.

Efface your own activities and promote those of the children as much as possible but always keep a watchful eye and a guiding hand upon them.

After the children have given number facts of their own choosing for a time

they are ready to begin memorizing in regular order the facts of addition and subtraction. For convenience these are divided into three sets. The first set consists of combinations in which both addends are even numbers. In the second set both addends are odd numbers. In the third set one of the addends is an even number and the other odd. For a detailed plan for teaching the first set of combinations and separations see page 18.

Observe that from this time on there are in each lesson two kinds of work, perception work and memorization, and that they are managed quite differently. In the perception work the children are not required to furnish from memory the facts needed in the exercises. They use their charts freely altho they are proud when they can say after finishing a piece of work, "I did it without my chart." But in this early memorizing work the children are to be led to depend upon the memory. For instance, when a class has learned to add 10, 2, and 4 to each even number on the chart, the teacher, before beginning that part of the day's lesson in which this particular set of facts is used in working examples or in tests, pins a large piece of paper over the wall chart and the children turn their desk charts over upon their desks. This is done as a necessary preliminary to the new and important kind of work, the memorizing. In case of a failure there is quick reference to the chart, but the ideal of successful work is that of sure (and later quick) memory work. The perception work gives pleasant and thoro preparation for the conscious, definite progress which is the real aim of all these efforts. Gradually as its purpose is accomplished it is dropped.

Until the first set of additions and subtractions are so well mastered that every child, except those who are to be retarded, has "stood test" upon them and can use them accurately, the children do not begin to memorize those of the second set, usually not before the third grade. If a child fails upon a test which he passed triumphantly a few weeks before, do not blame yourself, nor your predecessors, nor the child. The constitution of the ordinary human mind is such that there must be many temporary memorizations, many forgettings and renewings, before our knowledge of mathematical facts is permanent and readily available. It is the business of the school to furnish opportunities for these renewings, without haste or impatience, simply dealing with the mathematical nature of the child as it actually is, instead of assuming that it is what one might wish it to be.

As helps in the conscious memorizing we use the Identification Game, the Testing Game, played by partners, and Individual Tests. For these exercises, cards must be prepared. Flash cards, which are very useful, can be bought. As the list of successful test-passers grows, call for volunteers to try the speed test with cards or to be "it" in the game of "Catch me if you can."

Thruout the year, work and play with the multiplication and division tables is carried on parallel with that of addition and subtraction. One line of effort is stressed for a week or two, or until certain results appear, and then the other.

Knowledge of the table of tens is a great help in learning the table of fives. The children sing the table of fives, pointing out the multiples, or they sing and march. They make five-pointed stars out of pieces of colored paper, each of which is a symmetrical fifth of such a star. Reversing the little trapeziums that form the stars, they make ten-pointed figures. They are led to see and report such facts as that when they have twenty star points they can make four of the five-pointed stars or two of the stars that have ten points. Building oblongs with inch squares, making fans with toothpicks on the desks, reckoning pansy-petals (or those of any other five-petaled flower), are useful exercises. Pupils use the multiple squares, build the table on a tagboard or show it on a buttonboard, or play "Greeting the Multiples." (We use the word "multiple" in its objective sense, of course, and it is no more difficult for children than "automobile" or "radiator." To them it means "big, bright number on the chart.") They recite the table in order forward and backward, first in unison and then as individual "stunts". Then come the games with flash cards, the identification game and the

various kinds of tests. Many applications of the table of fives are made in number stories about nickels, dimes, etc. Some pupils like to play store, count money and make change and we offer them the chance, but that work goes better in the third grade. Making clock-faces and reckoning minutes by fives are useful exercises.

The same general plans are used in the teaching of all the tables, but each table has its own special applications. The table of twos gives many opportunities for reckoning pairs, as gloves, shoes, hands, feet, eyes. The children pour water from pint bottles into quart bottles and back again and discuss the number of pints and quarts of imaginary milk necessary to supply certain numbers of imaginary families. Before a new table is taken up the preceding ones are reviewed. The table of elevens (See page 15) is shown to the pupils as a flight of stairs down which they first walk, then run. It is very easily learned because pupils soon see such facts as that the fourth step is made of two fours, the seventh step of two sevens and so on. Here we begin the correlation of the tables by telling the children that there is in this table an "old friend" that we met in the five table, and setting them to hunt for it. When they have found 55 perhaps some one will report the finding of an old friend from the two table, 22.

In the last part of the second year pupils begin progressive written work with individual advance. This is a favorite occupation. A set of lesson sheets carefully graded, containing work which they have memorized, is prepared. In the work upon these sheets all begin together upon Lesson 1. If, upon examining the papers, the teacher finds an error, she cuts it out. At the succeeding lessons the work of each child begins where the perfect work of his last writing ended. When he has finished Lesson 1, in perfection without using the chart, he is allowed to take Lesson 2, and so on. A record of each child's progress is kept. After a time the children become separated in their individual work, some perhaps reaching Lesson 6 or 7 while others are still working upon Lesson 2. This process of natural selection aids in the sectioning of the grade at the end of the term.

Third Year Forty minutes a day. The new work of this grade is the first memorizing of the remaining facts of addition and subtraction, also of the tables of nines, threes, eights, fours, sevens, and sixes.

The first work is the thoro reviewing and testing of the first set of combinations and separations and of the tables of tens, fives, twos and elevens. New applications of these tables are made, as in dramatizing and in the Battle Game and in the finding of new fractional parts of the oblongs and other geometrical figures. Written multiplication is taught in connection with these tables. At first examples are given where no carrying is involved, such as the multiplying of 222 or of 521 by 2, 3, and 4. Later the pupils take such examples as the multiplying of 115 or 251 or 502 by each of the numbers from 2 to 9 inclusive. Notice that the multiplicands contain only the figures 1, 2, 0, and 5, and require knowledge of only the tables that are being reviewed.

Written subtraction is taught as the finding of a missing addend, by the method so clearly explained on pages 37-40 of the California state elementary arithmetic.

In order to prevent the repetition of mistakes, see that each child in your class has his correction-book, made by folding a sheet of paper, and that whenever he makes an error in reciting, he immediately puts the correct statement into his book. Occasionally drill upon these statements. You will find probably that there are certain facts upon which many pupils are apt to fail. Of course you will emphasize these facts.

As soon as the children are able to read problems intelligently, a primary number book is used as a basis for progressive written work. Before the books are given out, while the lesson sheets are still in use, the teacher prepares her class for the new difficulties of the book by teaching its first subjects in an easy

conversational way. In this teaching and in the work with the book the pupil is free to consult his chart at any time when his memory fails to supply promptly the desired fact. By this habit of carefully referring instead of guessing, mistakes are avoided and the facts are fixed in the mind. While the pupils are writing, the teacher ought not to move around among them. She should sit quietly in front of her class near the board. If a pupil needs help he should come forward after obtaining permission and put his problem on the board. Encourage your pupils to work independently of you, as much as possible.

The same plans of "teaching ahead" with the whole class, of testing, and of writing with individual advance are used in the succeeding grades. Without this careful and successful preliminary teaching the work in the book becomes difficult and confusing.

In teaching the second set of combinations and subtractions, use the same general plan as in teaching the first set. Begin with 3 and use it as an addend with numbers whose unit figure is 1, then 3, 5, 7, 9. As the additions are learned practice the corresponding subtractions, the finding of missing addends. Give tests and applications. When the children generally are sure and prompt in adding 3 to the odd numbers (and not before) use 5 as an addend in the same way, then 7, then 9. Test and apply at every stage, making the pupils feel that accuracy and progress are the two delightful things in their work and that their progress depends upon their accuracy. The first and second sets together include all the group combinations that form even numbers. Practice these combinations thru 18 thoroly before taking up the third set. If the memorizing of the first and second set of additions has been well done, that of the third is comparatively easy. To each of the numbers 1, 3, 5, 7 and 9 practice adding 2 until the pupils neither blunder nor hesitate. In the same way and with repeated tests and applications use as addends 4, 6, and 8. As a review, reverse the process, and to each of the even numbers 2, 4, 6, 8, add first 3, then 5, 7, 9. These form the group combinations of the odd numbers thru 17, and they should be faithfully practiced.

The table of nines on the chart presents a long flight of steps to be ascended and also the short, easy flight made of 90 and 99. There are four "old friends" in this table and the children should be allowed to hunt for them. The square yard furnishes a special application for the table of nines. Draw a square yard on the board and divide it into square feet. Let the children renew it as needed. Use it for a basis for number stories and problems. The table of threes, "the little sisters of the nines", as was said by a child, has special applications in dramatizing, in the game of "Threes Out" and in measurements with feet and yards. Measure things easily accessible. Finding heights of pupils is an interesting exercise. The buying and selling of ribbon (paper) at 5 or 2 or 9 or 3 cents a yard furnishes many problems. In connection with the tables of eights and that of fours much work is done with circles. Each child has his circle, made of paper or pasteboard and at least six inches in diameter. This he folds or cuts into parts, as halves, fourths and eighths. The pupils learn to draw circles and to bisect and otherwise divide them. They find out by inspection how many fourths or eighths it takes to make a whole. These sectors and the symmetrical parts of the octagon, which the pupils are taught to construct and divide, give many useful illustrations of the tables of eights and that of fours. "16 ounces in a pound" is given with the eight table and "4 quarts in a gallon," shown by use of the measures, is used in the number stories about the fours. The relative lengths of half notes, quarter notes and eighth notes should be shown. While the table of sevens is being memorized the fact that seven days make a week gives many problems. Besides the usual work with rectangles, fractional parts, etc., there is much buying or selling at 7 cents a yard, pound or gallon. In learning the table of sixes the hexagon, the six-pointed star, and their symmetrical parts, all of which the children construct and use in many decorative forms, furnish special applications of the table.

Fourth Year Work in the elementary state textbook is begun and one-half of each period is devoted to written advance in the book.

Each table is reviewed with new applications, such as factoring, common multiples, small fractions, etc. The battle game which has been played in the learning of the tables is now used so as to correlate them and to give much rapid work in factoring and cancelation.

The word "ratio" is used interchangeably with "parts" until the meaning is clear. Then the children are told that "ratio" also means "times". Much drill is given upon the reciprocal ratios of numbers.

Long division is begun with divisors 11, 12, 110, 101, 120, 121, etc. After a review of the multiplication table of nines, divisors such as 91, 910, 911, 89, 891, etc., are used. Each table is reviewed and used in connection with long division. In both short and long division each figure of the quotient should be placed directly above the righthand figure of that part of the dividend which produced it.

In the number stories stress is laid upon gain and loss in small commercial dealings with which children are familiar. Simple work is given in U. S. money, with bills and accounts, and in surface and boundary measurements.

Fifth Year The work of the preceding year has been largely anticipatory to the subjects as presented in the latter half of the elementary textbook, and in the fifth year the order of presentation in the book is generally followed. Games, tests and general devices used in the previous grades are adapted for use in this grade. In connection with ratio and proportion many exercises in working to a scale are drawn from the departments of Manual Training and Domestic Science.

In the number stories there is much simple reasoning based upon the child's intuitions of number and upon his understanding of simple business affairs as interest, profit and loss, etc. Continued number stories (problems involving two steps) are given. Those processes of written work, the reason for which the child can be led to discover, are accounted for, as for instance, the addition and subtraction of fractions. Others, as the division of one fraction by another, are given simply as processes leading to desired results, no attempt being made to force knowledge of the underlying principles into the immature mind. Formal analyses, as given in model solutions, are shown as fine, logical ways of reasoning, to be carefully considered but not memorized without understanding, nor used in vain repetitions as a substitute for original thinking.

My dear students, the working out of these plans which I have merely sketched, the adjusting of the work to the abilities of individual children, the careful preparation in class for indivdual advance, the promoting of the self-activity of the pupils, the insistence upon accuracy and the preventing of mistakes, the systematic mastery of certain portions of a subject before others are attempted,—all these require on your part earnestness in the work and a determination to succeed in it that will cause you to give much energy and to spend many hours in serious, fruitful thinking about your pupils' development. While grasping the general plan of this elementary instruction, if you concentrate upon the little span of work allotted to you, laboring intelligently, faithfully, and lovingly with the little ones entrusted to your care, you will have great rewards. Among these are the pleasure of seeing the success and happiness of your pupils, and also the consciousness of your own growing powers as a teacher. Hoping that these and other rewards may be yours, I am

Sincerely your friend and co-worker,

Adelia R. Hornbrook.

San Jose, Cal., Sept., 1914.

IV.

DRILLS

COMBINATION CHART

										No. of New Comb.
Ones	$\begin{array}{c}1\\\underline{1}\end{array}$	$\begin{array}{c}2\\\underline{1}\end{array}$	$\begin{array}{c}3\\\underline{1}\end{array}$	$\begin{array}{c}4\\\underline{1}\end{array}$	$\begin{array}{c}5\\\underline{1}\end{array}$	$\begin{array}{c}6\\\underline{1}\end{array}$	$\begin{array}{c}7\\\underline{1}\end{array}$	$\begin{array}{c}8\\\underline{1}\end{array}$	$\begin{array}{c}9\\\underline{1}\end{array}$	9
Twos	$\begin{array}{c}1\\2\end{array}$	$\begin{array}{c}2\\\underline{2}\end{array}$	$\begin{array}{c}3\\\underline{2}\end{array}$	$\begin{array}{c}4\\\underline{2}\end{array}$	$\begin{array}{c}5\\\underline{2}\end{array}$	$\begin{array}{c}6\\\underline{2}\end{array}$	$\begin{array}{c}7\\\underline{2}\end{array}$	$\begin{array}{c}8\\\underline{2}\end{array}$	$\begin{array}{c}9\\\underline{2}\end{array}$	8
Groups making Ten					$\begin{array}{c}9\\1\end{array}$	$\begin{array}{c}8\\2\end{array}$	$\begin{array}{c}7\\\underline{3}\end{array}$	$\begin{array}{c}6\\\underline{4}\end{array}$	$\begin{array}{c}5\\\underline{5}\end{array}$	3
Fives	$\begin{array}{c}1\\5\end{array}$	$\begin{array}{c}2\\5\end{array}$	$\begin{array}{c}3\\\underline{5}\end{array}$	$\begin{array}{c}4\\\underline{5}\end{array}$	$\begin{array}{c}5\\5\end{array}$	$\begin{array}{c}6\\\underline{5}\end{array}$	$\begin{array}{c}7\\\underline{5}\end{array}$	$\begin{array}{c}8\\\underline{5}\end{array}$	$\begin{array}{c}9\\\underline{5}\end{array}$	6
Groups making Five								$\begin{array}{c}4\\1\end{array}$	$\begin{array}{c}3\\2\end{array}$	0
Groups making Eleven					$\begin{array}{c}10\\1\end{array}$	$\begin{array}{c}9\\2\end{array}$	$\begin{array}{c}8\\\underline{3}\end{array}$	$\begin{array}{c}7\\\underline{4}\end{array}$	$\begin{array}{c}6\\5\end{array}$	2
Nines	$\begin{array}{c}1\\9\end{array}$	$\begin{array}{c}2\\9\end{array}$	$\begin{array}{c}3\\\underline{9}\end{array}$	$\begin{array}{c}4\\\underline{9}\end{array}$	$\begin{array}{c}5\\9\end{array}$	$\begin{array}{c}6\\\underline{9}\end{array}$	$\begin{array}{c}7\\\underline{9}\end{array}$	$\begin{array}{c}8\\\underline{9}\end{array}$	$\begin{array}{c}9\\\underline{9}\end{array}$	6
Groups making Nine						$\begin{array}{c}8\\1\end{array}$	$\begin{array}{c}7\\2\end{array}$	$\begin{array}{c}6\\\underline{3}\end{array}$	$\begin{array}{c}5\\4\end{array}$	1
Doubles	$\begin{array}{c}1\\1\end{array}$	$\begin{array}{c}2\\2\end{array}$	$\begin{array}{c}3\\\underline{3}\end{array}$	$\begin{array}{c}4\\\underline{4}\end{array}$	$\begin{array}{c}5\\5\end{array}$	$\begin{array}{c}6\\\underline{6}\end{array}$	$\begin{array}{c}7\\\underline{7}\end{array}$	$\begin{array}{c}8\\\underline{8}\end{array}$	$\begin{array}{c}9\\9\end{array}$	5
Eights	$\begin{array}{c}1\\8\end{array}$	$\begin{array}{c}2\\8\end{array}$	$\begin{array}{c}3\\8\end{array}$	$\begin{array}{c}4\\\underline{8}\end{array}$	$\begin{array}{c}5\\8\end{array}$	$\begin{array}{c}6\\\underline{8}\end{array}$	$\begin{array}{c}7\\\underline{8}\end{array}$	$\begin{array}{c}8\\8\end{array}$	$\begin{array}{c}9\\8\end{array}$	3
Groups making Eight						$\begin{array}{c}7\\1\end{array}$	$\begin{array}{c}6\\2\end{array}$	$\begin{array}{c}5\\3\end{array}$	$\begin{array}{c}4\\4\end{array}$	0
Other New Groups								$\begin{array}{c}4\\\underline{3}\end{array}$	$\begin{array}{c}7\\\underline{6}\end{array}$	2
							Total			45

LAWS OF COUNTING.

Counting by Twos

1. When the counting begins with two or any even number, the even numbers are named in order: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, etc.

2. When the counting begins with one or any odd number, the odd numbers are named in order: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, etc.

Counting by Tens

The unit figure is unchanged and the tens figure increases one each time: 11, 21, 31, 41, etc.

Counting by Fives

The alternate unit figures are the same: 1, 6, 11, 16, 21, 26, etc.

Counting by Elevens

Count forward ten and one; 1, 12, 23, 34, 45, etc. Note that the unit and tens figures are each increased one except when the unit figure is nine.

Counting by Nines

Count forward ten and backward one: 7, 16, 25, 34, 43, etc. Note that the unit figure decreases one, and the tens figure increases one each time except when the unit figure is 0. Before counting by nines practice counting backward by ones from 9 to 0, till it can be done without effort or mistake.

Counting by Eights

Count forward ten and backward two: 9, 17, 25, 33, 41, 49, etc., and 8, 16, 24, 32, 40, 48, etc. Note that when the counting begins with an even number, the even numbers are named in reverse order in the units place, and when it begins with an odd number, the odd numbers are repeated in reverse order. Before taking up counting by eights, practice counting backward by twos from 8 to 0 and from 9 to 1.

Laws similar to those given above govern counting by twelves, thirteens, nineteens, twenties, twenty-ones, etc. The pupil should be encouraged tho not required to discover and use them.

FIXING THE ADDITION-SUBTRACTION COMBINATIONS.

The addition-subtraction combinations should be well learned before the pupil is required to use them miscellaneously. This requires much repetition, and demands tact and patient perseverance on the part of the teacher. The following exercises will be found helpful:

1. Counting Exercise

This exercise is a kind of exposure, and may or may not result in the mastery of a given set of combinations. With the ones, twos, and tens it will usually be found sufficient.

2. Column Addition

At first the column should contain only the particular number under consideration, except at the foot, where any desired number may be placed. Later any whose combinations have been previously learned may be placed in the column. See that results only are named.

3. Decade Drill

For example:

$$\begin{array}{rrrrrl} 2 & 2 & 2 & 2 & 2 & \\ 3 & 13 & 23 & 33 & 43, & \text{etc.,} \\ \hline \end{array}$$

then:

$$\begin{array}{rrrl} 3 & 3 & 3 & \\ 2 & 12 & 22 & \text{etc.} \\ \hline \end{array}$$

In this work have the numbers and the results named: 3 and 2 are 5, 13 and 2 are 15, etc. Make use of the number chart: Name a number 2 greater than 6, 16, 36, etc.; 2 less than 37, 17, 67, etc.

4. Flash Drill

For this purpose the teacher should have a set of cards containing the forty-five combinations and blanks.

Hold a card before the class for a moment, then require some pupil to give the result.

OUTLINE FOR DRILL WORK FOR SECOND AND THIRD YEARS.

The ones in addition and subtraction will almost certainly be learned thru the counting exercise. Test and make sure by drill exercises.

1. Count by twos beginning with two and one; then beginning with any number.

2. Add single columns of numbers consisting of twos with any number at the bottom:

					2
				2	2
			2	2	2
		2	2	2	2
	2	2	2	2	2
2	2	2	2	2	2
7	7	7	7	7	7
—	—	—	—	—	—

Observe that each successive column repeats all the preceding ones. Later put ones among the twos and add. When 1 stands above another 1, two should be added.

3. Supply omissions involving twos:

( )	4	2	5	( )
2	( )	( )	( )	2
—	—	—	—	—
9	6	11	7	10

Give results as follows: 2 and 7 are 9; 4 and 2 are 6; 2 and 9 are 11, etc.

Later put the work in regular form for subtraction:

5	8	11
—2	—2	—9
—	—	—

Read 2 and 3 are 5; 2 and 6 are 8; 9 and 2 are 11.

4. Count by tens beginning with 10. Then beginning with 1, 2, 3, etc.

5. Name a number 10 more than a given number, 10 smaller; e. g., name a number ten more than 75, 43, 28, etc. 10 less than 96, 54, 87, etc. Use the number chart.

6. Learn the number groups making 10:

9	8	7	6	5
1	2	3	4	5

Use these groups in columns placing any number at the bottom. Later put twos and ones in the columns.

Supply omissions:

( )	7	8	( )	( )
6	( )	( )	9	5
—	—	—	—	—
10	10	10	10	10

Use flash cards. Keep the combinations learned on the board.

7. Count by fives beginning with 5. Then beginning with 1, 2, 3, etc. Note the law and compare it with counting by tens.

8. Add columns consisting of fives with any number at the bottom. Later mix in ones, twos, and groups making 10.

9. Learn the groups making five. Use these in column addition.

10. Supply omissions involving fives, ones, twos, and groups making 10. Later put in regular subtraction form.

11. Count by twos using the numeral frame or two rows of objects.

12. Learn the doubles to 9 and 9:

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

13. Make use of decade drills to fix these and other combinations studied:

6	6	6	6	
6	16	26	36	etc.

14. For written work fill the blanks in the following:

	+5	—5	+2	—2
27				
41				
34				
16				
43				
59				
65				
88				
92				

15. Teach the multiplication table of the ones, twos, tens, and fives in the order named. This work will be comparatively easy for the children for they have the foundation work already. The ones will require little if any drill. Cards containing one combination on each side will be found helpful.

(1) Count by twos beginning wtih two.

(2) Build the twos in multiplication as follows:

								2
							2	2
						2	2	2
					2	2	2	2
				2	2	2	2	2
			2	2	2	2	2	2
		2	2	2	2	2	2	2
	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2
—	—	—	—	—	—	—	—	—
2	4	6	8	10	12	14	16	18

This will enable the pupil to get at the meaning of the table. Similar building of the table may be continued in the tens and fives. It is an unnecessary burden to carry it further, for by that time the child will see that the table is based on counting by the given number beginning with the number itself.

(3) Write the twos in table form;

$1 \times 2 = 2$
$2 \times 2 = 4$
$3 \times 2 = 6$
$4 \times 2 = 8$ etc.

(4) Drill on the twos using drills and forms similar to those given for addition and subtraction also using cards.

(5) Write the twos in division and give drills as suggested above.

(6) Give exercises with the tens similar to those outlined for the twos. After the table has been written draw a line as here shown and have the pupils note that the right hand digit is always 0 and that the others are the same as the multiplier. (Do not be too ready to point out laws to the children. Give them time and reward their discoveries.):

$1 \times 10 = 1 \mid 0$
$2 \times 10 = 2 \mid 0$
$3 \times 10 = 3 \mid 0$
$4 \times 10 = 4 \mid 0$
$5 \times 10 = 5 \mid 0$

(7) Take up the fives in a similar manner. Write the table as shown:

$1 \times 5 = 5$
$2 \times 5 = 10$
$3 \times 5 = 15$
$4 \times 5 = 20$ etc.

Then separate it as follows:

$2 \times 5 = 1 \mid 0$	$3 \times 5 = 1 \mid 5$
$4 \times 5 = 2 \mid 0$	$5 \times 5 = 2 \mid 5$
$6 \times 5 = 3 \mid 0$	$7 \times 5 = 3 \mid 5$
$8 \times 5 = 4 \mid 0$	$9 \times 5 = 4 \mid 5$
$10 \times 5 = 5 \mid 0$	

Have the pupils note that when the multiplier is an even number the right hand digit is 0 and the other is half the even number. Why is this? When the multiplier is an odd number the right hand digit is 5 and the other is half the even number next smaller than the multiplier. For other drills, see directions given under multiplication and division.

16. Count backward by ones from 9 to 0. Repeat the exercises till it can be done quickly and accurately.

17. Count by nines beginning with 9, 1, 2, 3, etc. Note the law. Make use of the number chart.

18. Teach the nines in addition and subtraction. Use a table, and decade drills.

19. Add columns consisting of nines with any number at the bottom. Later place in the columns ones, twos, fives and groups making one, two, ten, or five.

20. Supply omissions involving nines and other combinations learned. Also use the regular subtracting form. Use cards containing combinations studied and omissions.

21. Learn the groups making nine

8	7	6	5
1	2	3	4
—	—	—	—

Use these in column work; fix them by decade drills and other exercises.

22. Count by elevens beginning with 11, 1, 2, 3, etc. Note the law. Use chart.

23. Learn the groups making eleven:

10	9	8	7	6
1	2	3	4	5
—	—	—	—	—

Use these in column work and in subtraction.

24. Count backward by twos from 8 to 0 and from 9 to 1. Repeat the exercise until it is mastered.

25. Count by eights beginning with 8, 1, 2, 3, etc. Note the law. Use chart.

26. Learn the eights in addition. Use the chart, decade drills and cards.

27. Add columns involving eights. Also supply omissions and subtract.

28. Review the groups making eight:

7	6	5	4
1	2	3	4
—	—	—	—

Use them in column work and in subtraction.

29. Learn the groups, $\begin{matrix}4\\3\end{matrix}$ $\begin{matrix}7\\6\end{matrix}$, and fix them by the necessary drills.

30. Review the forty-five combinations by groups. (See Group Chart.) Encourage the addition of two figures at a time in the group work. Where the sum of the two is more than ten refer it to ten or twenty. Thus:

$$\begin{array}{r}6\\7\\\hline 13\end{array}$$

13, is three more than 10, and the result when 6 and 7 are added together in a column is three more than the next higher decade. When

$$\begin{array}{r}9\\9\\\hline 18\end{array}$$

18, is added the result is two less than the second next higher decade.

Along with this drill work there should be woven in much concrete work. The problems should arise as far as possible from the pupils' employment at school, at home, at play. The actual weights and measures in the hands of the pupils will furnish material for problems, which should always be related to the neighborhood occupations and interests.

MULTIPLICATION AND DIVISION.

The ones, twos, tens and fives have already been studied. They should now be reviewed.

1. Count by elevens beginning with 11. Write the table of elevens in multiplication. Fix this by drills. The simple law and the rhythm make the learning of the elevens easy.

2. Count by nines beginning with 9.

3. Write the nines in Multiplication:

$$\begin{array}{l}
1 \times 9 = |9\\
2 \times 9 = 1|8\\
3 \times 9 = 2|7\\
4 \times 9 = 3|6\\
5 \times 9 = 4|5\\
6 \times 9 = 5|4\\
7 \times 9 = 6|3\\
8 \times 9 = 7|2\\
9 \times 9 = 8|1
\end{array}$$

Draw a line between the units and tens of the product. Note how the figures run in each column. The tens figure represents a number one less than the multiplier. The sum of the units figure and tens figure is nine. These facts will aid the pupil at first in recalling the nines. He must finally become able to recall a product immediately.

Of the forty-five combinations of the multiplication table up to 9 × 9, thirty have been studied. If taken up in order there will be five new combinations in the threes, four in the fours, three in the sixes, two in the sevens, and one in the eights. These and also the twelves should now be learned.

To memorize the multiplication table was thought too great a task for the child a few centuries ago.

and he was required to carry a box containing strips on which the ones, twos, threes, etc., in multiplication were written. When he was required to multiply by six, for instance, he took out the six-strip and referring to it found the products. It is not surprising then that children forget their tables, hesitate, and make mistakes. Much drill and much time are necessary.

Drill Work for Multiplication and Division.

1. Have the children write their own tables.

2. Have these tables repeated orally by each member of the class, looking at the table first, without looking at it later.

3. Have the tables written in tabular form as shown below:

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
—	—	—	—	—	—	—	—	—

This may be done as a class exercise on the board or paper. After the upper row has been written the other rows may be written in any desired order. The teacher may dictate a row thus:—Write two times seven, five times seven, seven times seven, four times seven, etc. The upper row may be written in any order and the class required to fill in the other rows as a seat exercise.

4. Take a given number and write its factors in sets of two. Thus:

$$24 = 6\times4 = 3\times8 = 2\times12$$

Use in this way all the products embraced in the multiplication table.

5. Find the numbers in each decade that a given number will divide. Take 8 for example: 1 to 10, $8 = 1\times8$; 11 to 20, $16 = 2\times8$; 51 to 60, $56 = 7\times8$, etc.

6. Find the nearest number not larger than a given number which another number will divide. Take 9 for example: 43, $36 = 4 \times 9$; 57, $54 = 6 \times 9$; etc.

7. Write the numbers from 0 to 12 in a vertical column in any desired order. Multiply any number up to 12 by each of these. This may be given as a seat exercise:

	$\times$ 6	$\times$ 8	$\times$ 4
3			
6			
8			
2			
5			
9			
12			
4			
7			
11			
1			
0			

Then erase the left hand column, place the sign of division before one of the numbers at the top, and restore the first column. Vary the form as suggested under addition and subtraction.

8. Use cards as in addition and subtraction. There are forty-five combinations in multiplication to 9×9, and eighty-one in division to $81 \div 9$, hence sixty-three cards will be required.

These drills should be given frequently even after the tables are supposed to have been learned.

For the work of the first three years the teacher should be supplied with a set of addition-subtraction cards (63), a set of multiplication-division cards (63), a good primary arithmetic containing many drill exercises, suggestions and illustration for concrete problems. The school should be supplied with balance scales; weights, one ounce to two pounds; foot ruler; yard-stick; pint, quart, half-gallon and gallon measures; clock face with movable hands; sets of cards containing addition-subtraction exercises, sufficient to supply each member of a class with a card of either set; large number chart containing numbers from 1 to 100 written in columns of tens, and quantities of inch cubes and inch squares.

GROUP CHART

$$2 = \begin{array}{r}1\\ \underline{1}\end{array}$$

$$3 = \begin{array}{r}2\\ \underline{1}\end{array}$$

$$4 = \begin{array}{r}3\\ \underline{1}\end{array}\quad \begin{array}{r}2\\ \underline{2}\end{array}$$

$$5 = \begin{array}{r}4\\ \underline{1}\end{array}\quad \begin{array}{r}3\\ \underline{2}\end{array}$$

$$6 = \begin{array}{r}5\\ \underline{1}\end{array}\quad \begin{array}{r}4\\ \underline{2}\end{array}\quad \begin{array}{r}3\\ \underline{3}\end{array}$$

$$7 = \begin{array}{r}6\\ \underline{1}\end{array}\quad \begin{array}{r}5\\ \underline{2}\end{array}\quad \begin{array}{r}4\\ \underline{3}\end{array}$$

$$8 = \begin{array}{r}7\\ \underline{1}\end{array}\quad \begin{array}{r}6\\ \underline{2}\end{array}\quad \begin{array}{r}5\\ \underline{3}\end{array}\quad \begin{array}{r}4\\ \underline{4}\end{array}$$

$$9 = \begin{array}{r}8\\ \underline{1}\end{array}\quad \begin{array}{r}7\\ \underline{2}\end{array}\quad \begin{array}{r}6\\ \underline{3}\end{array}\quad \begin{array}{r}5\\ \underline{4}\end{array}$$

$$10 = \begin{array}{r}9\\ \underline{1}\end{array}\quad \begin{array}{r}8\\ \underline{2}\end{array}\quad \begin{array}{r}7\\ \underline{3}\end{array}\quad \begin{array}{r}6\\ \underline{4}\end{array}\quad \begin{array}{r}5\\ \underline{5}\end{array}$$

$$11 = \begin{array}{r}9\\ \underline{2}\end{array}\quad \begin{array}{r}8\\ \underline{3}\end{array}\quad \begin{array}{r}7\\ \underline{4}\end{array}\quad \begin{array}{r}6\\ \underline{5}\end{array}$$

$$12 = \begin{array}{r}9\\ \underline{3}\end{array}\quad \begin{array}{r}8\\ \underline{4}\end{array}\quad \begin{array}{r}7\\ \underline{5}\end{array}\quad \begin{array}{r}6\\ \underline{6}\end{array}$$

$$13 = \begin{array}{r}9\\ \underline{4}\end{array}\quad \begin{array}{r}8\\ \underline{5}\end{array}\quad \begin{array}{r}7\\ \underline{6}\end{array}$$

$$14 = \begin{array}{r}9\\ \underline{5}\end{array}\quad \begin{array}{r}8\\ \underline{6}\end{array}\quad \begin{array}{r}7\\ \underline{7}\end{array}$$

$$15 = \begin{array}{r}9\\ \underline{6}\end{array}\quad \begin{array}{r}8\\ \underline{7}\end{array}$$

$$16 = \begin{array}{r}9\\ \underline{7}\end{array}\quad \begin{array}{r}8\\ \underline{8}\end{array}$$

$$17 = \begin{array}{r}9\\ \underline{8}\end{array}$$

$$18 = \begin{array}{r}9\\ \underline{9}\end{array}$$

ADDITION OF TWO OR MORE COLUMNS.

There should be no serious difficulty in teaching addition. A little reflection will convince the pupil that only units of the same order can be added.

Care should be taken that results only are named, and that when the sum of any column is 10 or more, the tens of the sum are combined with the first number added in the next column. Encourage the addition of two numbers at a time in groups, and insist upon it when the group sum does not exceed 11.

$$\begin{array}{r}6234\\ 2857\\ 1975\\ 7125\\ \hline 18191\end{array}$$

Say 10, 21; 4, 11, 19; 11, 21; 10, 18.

Should the pupil experience serious difficulty, let each column be added and the results combined. Such work is not uncommon in practice when long columns are to be added.

6234
2857
1975
7125
———
21 units
17 tens
20 hundreds
16 thousands
———
18191

Addition is a fatiguing exercise, and it is best that short columns be used at first. Keep the Group Chart before the class.

SUBTRACTION.

Subtraction is the finding of one addend in addition when the sum and the other addend are given, and should be approached thru supplying omissions in addition:

26354	( )
42641	42641
68995	68995

Say 1 and **4** are 5
4 and **5** are 9
6 and **3** are 9
2 and **6** are 8

8496	8496
5376	
13872	13872

Say 6 and **6** are 12
10 and **7** are 17
5 and **3** are 8
8 and **5** are 13

A similar process may be followed when there are more than two addends:

3748	
9452	9452
6187	6187
19387	19387

Say 9 and **8** are 17
6, 14 and **4** are 18
6 and **7** are 13
10, 16 and **3** are 19

The usual form may then be used.

8769
4532
———
4237

Say 2 and **7** are 9
3 and **3** are 6
5 and **2** are 7
4 and **4** are 8

14369
6548
———
7821

Say 8 and **1** are 9
4 and **2** are 6
5 and **8** are 13
7 and **7** are 14

It is well that all should subtract in this manner, but not wise to compel those who subtract by a different method to change. The teacher, however, should make this her habitual method of subtracting. It is not necessary that the subtrahend should be written below the minuend.

Teach the business way of making change. For example, if the sale amounts to $1.65 and $5.00 is given in payment, the salesman would say $1.75, $2.00, $3.00, $4.00, $5.00, laying down in succession 10c, 25c, $1.00, $1.00 and $1.00. The change is taken from the cash register in the same order.

MULTIPLICATION

Multiplication presents no serious difficulty after the tables have been learned; and these must be mastered. If, however, a pupil is required to multiply by 8 and does not know the eight table, have him write it on a strip of paper and make reference to it when necessary. See drills for fixing the multiplication table.

Use one figure as a multiplier at first, and place the emphasis on the *how* rather than the *why*.

The main difficulty in multiplying large numbers arises from the fact that children forget the tables. To aid pupils in recalling and fixing the multiplication table use the same multiplier till it is well learned, placing the emphasis on the tables, probably the 6's, 7's, and 8's, that are most difficult. The same suggestions apply in division using small divisors.

DIVISION.

Teach long division first, using divisors not exceeding twelve. See that the quotient is written above the dividend, each quotient figure being placed directly over the right hand figure of the dividend used in obtaining it. First multiply a number. and then reverse the process. For example:

```
     3647
        8
   ------
       56
      32
     48
    24
   ------
    29176
     3647
   ------
8 ) 29176
    24
    --
     51
     48
     --
      37
      32
      --
       56
       56
```

After the pupil has become able to divide by long division using small divisors, introduce short division as an abbreviation. Then require that short division shall be used for all divisors less than twelve.

```
     3647
   ------
8 ) 29176
```

Do not permit the pupil to write the remainders, or to see any one else write them.

Long Division With Large Divisors.

Long division when the divisors are large presents three difficulties to the beginner: the form, the approximation, and the testing. These should be overcome one at a time. The form should be mastered while using small divisors as suggested above.

Divide 298744 by 856 and then note the processes by which the result is obtained:

```
        349
    -------
856)298744
    2568
    ----
     4194
     3424
     ----
      7704
      7704
      ----
```

First we say 8 in 29 three times. Then we try 3 for a quotient figure, and find that after dividing 29 by 8 we have 5 as a remainder, which placed with the 8 makes 58, a number more than sufficient to contain 5 three times. Hence the first quotient figure is 3 and the next dividend to be used is found to be 4194. 8 in 41 five times with a remainder of 1, which being joined with nine gives 19, a number which will not contain 5 five times. Hence the next quotient figure is not 5 but 4. So we continue to approximate and test, using first the left hand figure then testing, using one or more of the succeeding figures of the divisor.

To master this approximation requires much experience on the part of the child and time for growth. A rule, however explicit, will not suffice.

1. Use such divisors as 20, 30, 40, 50, 60, etc. Then the left hand figure may be used as a trial divisor and no testing is needed.

2. Use divisors in which the second figure is one: as 41, 61, 713, 816, etc. The left hand figure is used as a trial divisor, and the result will nearly always be correct.

Place a number of dividends on the board and have pupils find and test quotient figures by inspection, using divisors like 21, 31, 41, etc. Repeat the exercises from day to day.

3. Use divisors in which the second figure is 2. When the left hand figure is used as a trial divisor the result will need more careful testing and correcting.

Place several dividends on the

board, and find and test the quotient figure using 22, 32, 427, 521, etc., as divisors.

)68)84)122)158)198

4. Use divisors in which the second figure is 9. The number next larger than the left hand may then be used as a trial divisor and the result tested and corrected.

Continue the placing of dividends on the board for inspection tests.

In this way the pupil will gradually learn how to find and test the quotient figure. The teacher, however, must be satisfied with slow progress, must persevere, and be patient.

FRACTIONS.

Children will almost certainly know something of fractions when they enter school. What child has not had to share his apple, his orange, or his marbles. There can be no valid objection against the introduction of fractions with small denominators in the early years of the course. These fractions should be concrete: as 3 fourths of a gallon, 2 thirds of a foot, etc., and the denominator should be expressed in words, and thought of as the names of the parts. Similar fractions may be added or subtracted, or a fraction may be multiplied or divided by a whole number. No rules should be given. A more extended study of fractions should be taken up in the fifth year, and a fuller treatment be given during the sixth year and later.

REDUCTION.

1. Reducing improper fractions to whole or mixed numbers and whole or mixed numbers to improper fractions presents no serious difficulty.

2. Reducing to higher terms may be illustrated by taking two halves of an apple and dividing each into 2 or 3 parts. It may then be seen that $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$, etc

A better way, however, is to furnish the pupils with strips of paper of uniform length and have these folden as follows:

a. Lay one strip down without folding.

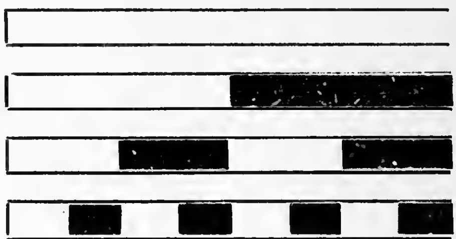
b. Fold one strip and crease it in the center.

c. Fold a third strip and crease it into fourths.

d. Fold a fourth strip and crease it into eighths.

Place these strips side by side, and it will readily be seen that
$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$; $\frac{1}{4} = \frac{2}{8}$; $\frac{3}{4} = \frac{6}{8}$

e. Draw diagrams on the board representing these divisions.



In a similar manner diagrams may be drawn representing the relations of halves, thirds, and sixths; of halves. thirds, fourths, sixths, twelfths, etc.

The divided apple is concrete. The creased paper or the drawing is a representative concrete and becomes a tool which the pupils may use or image in determining other fractional relations. Dealing with fifths, for instance, it will be found that

$\frac{2}{5} = \frac{4}{10} = \frac{8}{20} = \frac{10}{25}$, etc.

Observing these and other fractional equivalents, which may be worked out, the pupil will get and understand the rule. He will also see that fifths cannot be reduced to ninths, fourths to elevenths, etc.

3. Reducing fractions to lower terms follows as a corollary from number two and needs no illustration.

4. Reducing fractions to a common denominator should be first introduced in connection with addition and subtraction. It should later be studied as a separate topic. Small

fractions should be given as a rule and the work done principally by inspection. If, because of the largeness of the common denominator, written calculations are required, let these be done as side work. The least common denominator is the l. c. m. of the denominators and should be found as set forth under that topic.

ADDITION AND SUBTRACTION

There is no special difficulty in teaching addition and subtraction of fractions, but care should be taken that good forms are used.

1. Give much drill in adding and subtracting similar fractions. This work may be introduced along with addition and subtraction of whole numbers provided the denominator is expressed in words. Write fractions to be added or subtracted under each other, especially when there are mixed numbers.

$$\begin{aligned} 6\tfrac{3}{4} &= 6\tfrac{9}{12} \\ 7\tfrac{1}{2} &= 7\tfrac{6}{12} \\ 9\tfrac{1}{3} &= 9\tfrac{4}{12} \\ 12\tfrac{5}{6} &= 12\tfrac{10}{12} \\ \hline 36\tfrac{5}{12} &= 34\tfrac{29}{12} \\ (\tfrac{29}{12} &= 2\tfrac{5}{12}) \end{aligned}$$

2. Use full form for work in the fifth year writing the reduced answer directly under the first column. Use a similar form in subtraction. Write out in full when the fractional part of the subtrahend exceeds the fractional part of the minuend. Avoid the term "borrow." Take one from the 52 and change it to twenty-fourths.

$$\begin{aligned} 27\tfrac{5}{6} &= 27\tfrac{20}{24} \\ 19\tfrac{3}{8} &= 19\tfrac{9}{24} \\ \hline & 8\tfrac{11}{24} \end{aligned}$$

$$\begin{aligned} 52\tfrac{5}{8} &= 52\tfrac{15}{24} = 51\tfrac{39}{24} \\ 27\tfrac{2}{3} &= 27\tfrac{16}{24} = 27\tfrac{16}{24} \\ \hline & \qquad\qquad 24\tfrac{23}{24} \end{aligned}$$

3. Use the abbreviated form when addition and subtraction are studied in the sixth and subsequent years.

	16
$12\frac{1}{4}$	4
$27\frac{3}{8}$	6
$63\frac{7}{16}$	7
$103\frac{1}{16}$	$\frac{17}{16} = 1\frac{1}{16}$

	36
$83\ \frac{9}{12}$	21
$36\ \frac{2}{9}$	8
$47\frac{13}{36}$	$1\frac{3}{36}$

Add and reduce the fraction first, place the fraction of the result under the fraction of the addends, and add the integral part, if any, along with the other whole numbers.

ALIQUOT PARTS.

The customary way of presenting aliquot parts is to give a list of aliquot parts of 100 and require that they shall be memorized, then to give rules for the use of these numbers in multiplication and division. No effort is made to assist the pupil in memorizing the list, or to lead him to understand and appreciate the principle underlying the rules. Such work has no educational value and will benefit only the pupil who soon engages in some commercial business which makes use of it.

In what follows an effort is made to relate the work so as to aid the memory and so that laws for multiplication and division will be impressed.

1. Count by $2\frac{1}{2}$ beginning with $2\frac{1}{2}$. Continue the counting till a law is found. $2\frac{1}{2}$, 5, $7\frac{1}{2}$, 10, etc. Note that the ending are repeated in order in sets of four. Note also that $4 \times 2\frac{1}{2} = 10$, $8 \times 2\frac{1}{2} = 20$; in general that there will be as many tens in the product as there are fours in the multiplier. Conversely, when 10, 20, 30, etc., is divided by $2\frac{1}{2}$ the quotient will be as many times 4 as the number has tens.

Multiply $2\frac{1}{2}$ by 12, 28, 24, 36, 16, 32, 44, etc.

Divide 40, 70, 30, 50, 90, 60, 80 by $2\frac{1}{2}$.

2. Write and learn the table of $2\frac{1}{2}$ to $4 \times 2\frac{1}{2}$. If any number is multiplied by $2\frac{1}{2}$, the product will contain as many tens as the number has fours and the units will be $2\frac{1}{2}$,

5, or $7\frac{1}{2}$, according as there is a remainder of 1, 2, or 3. Thus $25 = 6 \times 4 + 1$, hence $25 \times 2\frac{1}{2} = 62\frac{1}{2}$; $35 = 8 \times 4 + 3$, hence $35 \times 2\frac{1}{2} = 87\frac{1}{2}$.

Any number which ends in 0, $2\frac{1}{2}$, 5, or $7\frac{1}{2}$ is exactly divisible by $2\frac{1}{2}$. To obtain the quotient, multiply the tens (all above units) by 4 and to the product add the quotient obtained by dividing the units by $2\frac{1}{2}$. $57\frac{1}{2} \div 2\frac{1}{2} = 5 \times 4 + 3 = 23$; $115 \div 2\frac{1}{2} = 11 \times 4 + 2 = 46$.

3. Deal with $3\frac{1}{3}$ in a similar manner and obtain laws for multiplication and division by $3\frac{1}{3}$. Give much exercise in the use of these laws.

4. Count by 25; learn the twenty-five table in multiplication to 4×25. Multiply and divide by 25 until such work becomes easy. Note the similarity of the 25 and $2\frac{1}{2}$ tables.

5. Count by $12\frac{1}{2}$ to 100. Place the results side by side with the results obtained by counting by $2\frac{1}{2}$ and 25.

		$12\frac{1}{2}$
$2\frac{1}{2}$	25	25
		$37\frac{1}{2}$
5	50	50
		$62\frac{1}{2}$
$7\frac{1}{2}$	75	75
		$87\frac{1}{2}$
10	100	100

Note that the unit figures agree with the units of the $2\frac{1}{2}$ series; also that every second result agrees with a result in the 25 series.

6. Write the $12\frac{1}{2}$ table in multiplication to $8 \times 12\frac{1}{2}$. Place the table aside for quick reference.

If a number is multiplied by $12\frac{1}{2}$, the product will contain as many hundreds as the number has eighths and the tens and units of the product will be the product of $12\frac{1}{2}$ and the remainder as shown in the table. $43 = 5 \times 8 + 3$, hence $43 \times 12\frac{1}{2} = 537\frac{1}{2}$; $77 = 9 \times 8 + 5$, hence $77 \times 12\frac{1}{2} = 962\frac{1}{2}$.

A number which ends in any number found in the table is exactly divisible by $12\frac{1}{2}$. The quotient is eight times the hundreds figure (all over tens) plus the quotient obtained by dividing the units and tens by $12\frac{1}{2}$. $637\frac{1}{2} \div 12\frac{1}{2} = 8 \times 6 + 3 = 51$; $1187\frac{1}{2} \div 12\frac{1}{2} = 8 \times 11 + 7 = 95$.

7. Take up $33\frac{1}{3}$, $16\frac{2}{3}$, and $8\frac{1}{3}$ in succession and deal with them in a similar manner, referring the results to the $3\frac{1}{3}$ series. If the series are placed side by side, the similarity and the laws will be emphasized. Select the new results in each series in order and learn them.

In this connection take up bills and accounts. Use regular business forms. Show the class samples of bills from different mercantile establishments. Have the pupils rule their own bills and fill them out in a neat and orderly manner. Encourage getting results by inspection. Let other necessary calculations be done as side work. Introduce receipts, drafts, and checks. It will be found that this touching of actual business forms and customs will be very attractive to the pupils.

V.

COURSE

FIFTH YEAR.

FRACTIONS.

Impress and deepen the fraction concept by divided apples, oranges and other objects; ruler, yard-stick, pint, quart, and gallon measures, pound and other weights, etc., and folded paper.

Change whole numbers and mixed numbers to fractions.

Change improper fractions to mixed numbers.

Add and subtract fractions having a common denominator. Add and subtract mixed numbers in a similar manner.

Factor products found in the multiplication table.

Find by inspection the common multiples of numbers. The common multiples should be the products found in the multiplication table. While the least common multiple should be sought, it is not vital that these terms should be singled out for special study.

Find by inspection common divisors of numbers to twenty, to thirty, to fifty. The numbers should be products found in the multiplication table, and two or not more than three numbers should be used at a time.

Reduce fractions to larger denominators, Use objects, folded paper, and drawings to illustrate.

Add and subtract fractions of different denominators using the full form. The common denominator should be limited to products found in the multiplication table.

Reduce fractions to their lowest terms.

Use cancellation where there is indicated multiplication and division.

Multiply a fraction by a whole number. Multiply a mixed number by a whole number. When the integral part is small the expression should be reduced to an improper fraction; when it is large the fractional part should be multiplied first. If the multiplier does not exceed twelve, which should be the rule at this time, the carrying should be done at once as in other multiplication.

Divide a fraction by a whole number. Divide a mixed number by a whole number. When the integral part is small reduce to an improper fraction; when it is large divide as in whole numbers and reduce only the remainder to a fraction. Use small divisors.

Find a fractional part of a whole number. Multiply a whole number by a fraction. Use cancellation. Multiply a whole number by a mixed number. Do not reduce to improper fractions unless the integral part is quite small.

Find a fractional part of a fraction. Multiply a fraction by a fraction. Give rule and use cancellation. Reduce mixed numbers to improper fractions.

Find a number when a fractional part of it is given. Divide a whole number by a fraction. Indicate the work and use cancellation.

Divide a fraction by a fraction. Use inversion, indicate the work, and cancel.

DECIMALS.

Introduce decimals by using problems requiring addition and subtraction of United States money. Add and subtract other decimals.

Multiply United States money by a whole number, and by a mixed number using forms for bills and other accounts. Multiply other decimals by whole or mixed numbers.

Divide United States money by a whole number. Divide other decimals by a whole number. Reduce common fractions to decimals. Place the quotient above the dividend, and place the decimal point in the quotient as soon as the decimal point in the dividend is reached.

Multiply a decimal by a decimal. Use the word per cent and the character, %, interchangeably with hundredths. Find any per cent of a number. Express the per cent decimally when using it. Find interest on money, confining the time to years, or years and months which will make easy fractions of years.

Divide a decimal by a decimal. First mark off as many decimal places in the dividend counting from the decimal point, as there are decimal places in the divisor. Place the decimal point in the quotient as soon as the mark in the dividend is reached.

SIXTH YEAR.

State Advanced Arithmetic.

It is assumed at this time that the pupil has an elementary knowledge of fractions, that he can add, subtract, multiply and divide decimals

and that he has done some simple work in percentage.

Review reading and writing of whole numbers and decimals to billions, by the Hindu-Arabic notation. Learn to count to one hundred by the Roman notation, learn the significance of I, V, X, L, C, D, and M. Learn the additive and subtractive laws. Express various dates in the Roman notation.

Recall and learn the tables for long measure (inches, feet and yards), liquid measure (gills, pints, quarts and gallons.)

Review addition and subtraction of whole numbers and decimals. In this review follow the group chart, and insist that two figures be added at once when their sum does not exceed 12. For example when $\begin{array}{c}2\\3\\\hline\end{array}$ or $\begin{array}{c}4\\1\\\hline\end{array}$ stand one above the other for addition, add 5. Count by 10, 11, 9, 12, and 8, emphasizing the law, and then add groups making one of these numbers. Give examples in compound addition and subtraction involving the tables learned.

Review multiplication of whole numbers and decimals. Give systematic and persistent drills on the multiplication tables, emphasizing the 6's, 7's and 8's. Impress the laws for 5's. Give exercises in multiplication of compound numbers. Take up special cases in multiplication: (1) Multiplying by one with any number of ciphers annexed; (2) Multiplying by any number with ciphers annexed; (3) Multiplying by 25 and $12\frac{1}{2}$; (4) Multiplying by $33\frac{1}{3}$, $16\frac{2}{3}$, $8\frac{1}{3}$ and $66\frac{2}{3}$.

Introduce bills and accounts. Use regular bill paper.

Find areas of rectangles and work out table for square measure for square inches, square feet, and square yards.

Find volume of rectangular solids and work out table for cubic inches, cubic feet, and cubic yards.

Review division of whole numbers and decimals. Use short division where the divisor does not exceed twelve and place quotient above or below the dividend as is most convenient. Use long division when the divisor exceeds twelve. When the divisor is a decimal mark off as many decimal places in the dividend as there are decimal places in the divisor. Place the decimal point in the quotient as soon as the mark in the dividend is reached.

Reduce a denominate number to units of a higher order.

Take up special cases of division, such as (1) Dividing by one with ciphers annexed; (2) Dividing by any number with ciphers annexed; (3) Dividing by 25 and $12\frac{1}{2}$; (4) Dividing by $33\frac{1}{3}$, $16\frac{2}{3}$ and $8\frac{1}{3}$.

Give some work in compound division using such divisors as 2, 3, 4, 5; also some inverse problems in rectangles and rectangular solids.

FACTORING.

Factor (1) Products in the multiplication table, writing two factors and then resolving these factors into prime factors; (2) Other numbers less than 100; (3) Numbers of three figures in which the right hand figure is 0.

Learn and apply the laws for divisibility: (1) Two, five and ten; (2) Four and twenty-five; (3) Nine, three and six.

COMMON DIVISORS.

Find common divisor by inspection of numbers under fifty.

Find the greatest common divisor of given numbers under two hundred by factoring one of them.

Apply common divisors in reducing fractions to their lowest terms, and in cancellation.

COMMON MULTIPLES.

Find common multiples of numbers by inspection, when such common multiples do not exceed fifty. Find the least common multiple by factoring, the l. c. m. not to exceed one hundred.

ADDITION AND SUBTRACTION OF FRACTIONS.

Add fractions using the abbreviated form, the common denominator being a product of the multiplication table; subtract fractions using the abbreviated form, the common denominator not to exceed fifty

MULTIPLICATION AND DIVISION OF FRACTIONS.

Multiply a fraction by a whole number, give rule and have it learned. Multiply a mixed number by a whole number; multiply a complex decimal by a whole number. Reduce denominate fractions to integers of lower orders.

Divide a fraction by a whole number, (a) when the numerator exactly contains the divisor; (b) When the numerator does not exactly contain the divisor. Give rule and have it learned. Divide a mixed number by a whole number. Divide a complex decimal by a whole number.

Multiply a whole number by a fraction, also by mixed number and a complex decimal. Multiply a fraction by a fraction, a mixed number by a fraction, a complex decimal by a fraction. Similarly multiply by a mixed number and by a complex decimal.

Divide a whole number by a fraction, a fraction by a fraction, a mixed number by a fraction, and a complex decimal by a fraction. Similarly divide by a mixed number and a complex decimal.

FRACTIONAL RELATIONS.

Find a required part of a number.

Find a number when a certain part of it is given.

Find what part one number is of another.

Apply the same to fractions.

SEVENTH YEAR.

PERCENTAGE.

Express per cent as a decimal and apply the same in finding required per cent of a number.

Express certain per cents as common fractions and apply the same in finding a required per cent of a number or fraction.

Find a required number when a certain per cent of it is given, first expressing the per cent decimally, or as a fraction.

Express a decimal as per cent; also change a common fraction to per cent.

Find what part one number is of another and find what per cent one number is of another.

Find a number a given part greater than another and find a number a given per cent greater than another.

Find a number a given part less than another; also a given per cent less than another.

Find a number when another number a given part or per cent greater than it is given; also when one given part or per cent less than it is given.

APPLICATION OF PERCENTAGE. INTEREST.

Find interest for years, years and months, and months and days.

Find the time by compound subtraction and by counting the exact number of days. In the latter case, reckon 360 days as a year for commercial transaction, and 365 for a year for exact interest.

Study commercial paper, including notes, drafts, checks and money orders.

Study banking, including bank discount and savings accounts.

EIGHTH YEAR.

Build up squares and extract square root. Study and solve right triangles.

SURFACES.

Land measure, including United States land divisions, land surveyor's chain. Vara.

Lumber measure, including shingling.

Plastering, papering and carpeting.

Areas of parallelograms, triangles, trapezoids and circles.

VOLUMES.

Rectangular solids, prisms and cylinders.

Pyramids and cones.

Frustums.

Spheres.

Problems in analysis and ratio and proportion.

Longitude and Time, including a discussion of Standard Time.

VI.

REVIEWS

1. READING AND WRITING NUMBERS.

HINDU-ARABIC NOTATION.

1. Numbers are usually expressed in writing by the use of ten characters called digits. These digits are 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, and standing alone represent—one, two, three, four, five, six, seven, eight, nine, and naught, respectively, and are so named.

In writing a number each digit represents a value dependent on the place which it occupies, this value increasing from right to left in a ratio tenfold. In expressing a whole number the right hand digit represents the same value as the digit when standing alone; when in the second place counting from the right it represents a value ten times as great; when in the third, one hundred times as great, and so on.

Thus, in 1111 the right hand digit represents one, the next ten, the next one hundred, and the next one thousand. In 22222 the right hand digit represents two, the next twenty, the next two hundred, the next two thousand, and the next twenty thousand. What does each digit represent in 163435?

For convenience in reading these numbers are grouped into periods of three figures each beginning with units, or the right hand digit in whole numbers. These groups are frequently separated by commas to aid the eye in distinguishing them.

The first five periods are named as follows beginning at the right: units, thousands, millions, billions and trillions. Each place is called an order, and beginning at the right the orders are named as follows: units, tens, hundreds, thousands, ten-thousands, hundred-thousands, millions, ten-millions, hundred-millions, etc.

Separate the following numbers into periods and name the periods and the orders: 476328, 765342708, 62840070903.

It should be noted that each period except the left hand one must have three places and that 0 is written in each vacant order.

2. Reading Numbers. In reading numbers the periods are read in order beginning with the one at the left. The word *and* is unnecessary in reading whole numbers and should be omitted. For example, 60,705,821,062 should be read sixty billion, seven hundred five million, eight hundred twenty-one thousand, sixty-two. 964,000,083 is read nine hundred sixty-four million, eighty-three.

Read the following numbers:

1.	671;	671,000;	671,000,000;
2.	403;	4,003;	40,300;
3.	520;	52,000;	5,200,000;
4.	89;	89,089;	8,900,890;
5.		47,407,047;	4,700,400,747;
6.		67,004;	149,820;

600,700,100.
400,003,000.
50,000,020.
890,089,809.
407,470,047,407.
74,000,937,506.

Name the order and give the value expressed by each significant digit in the following numbers, then read each number:

682,251; 403,075,209; 20,704,641

3. Writing Numbers. In writing numbers each period except the left one must be full, each vacant order being filled with 0.

Place the following on the paper or board and write the numbers under it:

Trill.	Bill.	Mill.	Thous.	Units
h.t.tr.	h.t.b.	h.t.m.	h.t.th.	h.t.u.
0 0 0,	0 0 0,	0 0 0,	0 0 0,	0 0 0

Write the following numbers, separating the periods by commas:

1. Two thousand six hundred fifty-three.
2. Six thousand sixty-six.
3. Four million four hundred thousand four.
4. Twenty-seven billion two hundred seven thousand two hundred seventy.
5. One hundred sixteen trillion eighty-four billion five hundred sixty-one million four hundred.
6. Four million 27 thousand 36.
7. Twelve billion 305 thousand.
8. Three hundred seventy-one trillion 60 million 8 thousand 406.
9. 200 billion 2 million 20 thousand 202.
10. 72 million 7 thousand 40.

Write the following, omitting the commas:

1. 600 thousand 25.
2. 902 million 3 thousand 800.
3. 47 billion 706 million 9.
4. 6 trillion 15 million 28.
5. 391 billion 408 thousand.
6. 560 trillion 38 billion 7 million 500 thousand.
7. 74 million 200 thousand 65.
8. 400 million 5 thousand 6.
9. 7 trillion 5 million 3 thousand 1.
10. 60 billion 800 million 4 thousand 20.

Read all the numbers which you have written.

FACTORING.

FACTORING BY INSPECTION.

Small numbers may be factored by inspection by first separating each number into two factors and factoring the factors when possible.

$$24=4\times6=2\times2\times2\times3$$
$$54=6\times9=2\times3\times3\times3$$
$$28=4\times7=2\times2\times7$$

In like manner factor the following numbers:—

1. 36, 42, 45, 32, 56, 63, 75, 81, 44, 48. It is not necessary that the pairs of factors be written. They may be thought and the final factors only need be written.

Factor the following numbers, writing the prime factors only:—

2. 64, 66, 84, 96, 108, 144, 88, 99, 27, 125.

Numbers which end in 0 contain the factor 10 or 2×5. In factoring such numbers the 2×5 should be written first, thus:

$$210=2\times5\times5\times7$$
$$3300=2\times5\times2\times5\times3\times11.$$

Any number less than 1000 ending in 0 can be readily factored by inspection.

3. Factor 240, 340, 360, 420, 720, 630, 640, 660, 490, 390.

4. Factor 800, 900, 1200, 1800, 1600, 3100, 44,000, 75,000, 84,000, 640,000.

DIVISIBILITY OF NUMBERS.

There are certain tests by means of which it may be quickly determined whether or not a number is divisible by certain other numbers, as 2, 5, 4, etc. These tests are all based on the fact that numbers are written on the scale of ten, i. e., that ten ones make ten, ten tens make a hundred, etc.

1. Two will divide a number if it ends in 0, 2, 4, 6, 8.

2. Five will divide a number if it ends in 0 or 5.

3. Ten will divide a number if it ends in 0.

If the right hand digit is 0 the number is made up of tens. 370=37 tens, 6430=643 tens. Since two, five, and ten each divides ten it will divide 37 tens or 370, 643 tens, or 6430. Two will likewise divide 370 plus 6 or 376, 6430 plus 8, or 6438. Five will divide 370 plus 5, or 375, and 6430 plus 5, or 6435.

Applying the above principles factor the following numbers:

162, 98, 126, 132, 176, 175, 165, 105, 1150, 1450.

4. Four will divide a number if it will divide the number expressed by its two right hand digits.

5. Twenty-five will divide a number if it ends in 00, 25, 50 or 75. Principles 4 and 5 depend upon the facts that four and twenty-five will each divide 100, and that if the number ends in 00 it is made up of hundreds.

Factor the following numbers by applying principles 4 and 5:

168, 232, 216, 288, 324, 375, 675, 825, 950, 625, 1125.

In factoring always use the form given at the beginning of this chapter. If it is necessary to take out a factor by division, do it by using side work.

$100\div25=4$. Hence $200\div25=8$, $300\div25=12$. $700\div25=7\times4=28$ $1500\div25=15\times4=60$.

It will be seen from these examples that when a number ending in 00 is divided by 25 the quotient may be obtained by omitting 00 and multiplying by 4.

Write the quotient obtained by dividing the following numbers by 25: 17600, 1300, 15200, 19000, 26000, 34700, 29600, 76500, 43900.

$1275=1200+75$; hence $1275\div25=48+3=51$. $2950=2900+50$; hence $2950\div25=4\times29+2=118$. $14775\div25=4\times147+3=591$.

Divide by 25: 775, 925, 1350, 5375, 6750, 4125, 43775, 54300, 27625, 49350.

6. Nine will divide a number if the sum of its digits is divisible by 9.

7. Three will divide a number if the sum of its digits is divisible by 3.

8. Six will divide an even number if the sum of its digits is divisible by 3.

Applying tests 6, 7 and 8, determine whether 3, 6 or 9 will divide the following numbers: 474, 684, 543, 856, 7485, 8694, 571, 735682, 88875, 77772.

Factor: 147, 693, 198, 234, 363, 251, 2430, 294, 396, 16200.

Whether or not a number is divisible by 7, 11, or 13 or any larger prime number is best determined by trial. If the prime numbers are tried in the order of their size it will never be necessary to try a prime number for a number less than the square of that prime number. If no factor is found the number is prime.

Try all the odd numbers not ending in 5 between 50 and 100. Write the factors of the composite ones.

Determine the primes and factor the composites: 343, 199, 187, 171, 299, 323, 209, 289, 371, 357.

GREATEST COMMON DIVISOR.

In order to find the greatest common divisor of two or more numbers one should first determine the prime factors of one of them. These factors may then be tried in the other numbers and those which are common determined. The product of the prime factors that are common will be the greatest common divisor.

Find the greatest common divisor of 84, 105, 147, 189.

$$84=\cancel{2}\times\cancel{2}\times\underline{3}\times\underline{7}$$

Therefore $3\times7=21$, G. C. D. First factor 84 as shown above. Then try these factors 2, 3, and 7 in the numbers 105, 147 and 189. By applying the tests for divisibility to 2 and 3, trying 7, it is found that 2 will not divide each of the num-

bers and 3 and 7 will. Cross out the 2's and underscore 3 and 7, and the product of 3×7, or 21, is the G. C. D.

Find the greatest common divisoı of the following sets of numbers:

1. 78, 130, 156, 208.
2. 70, 175, 210, 315.
3. 52, 91, 143, 221, 260.
4. 95, 133, 247, 437.
5. 99, 231, 451.

Sometimes the same factor is common two or more times. In such case divide each of the numbers except the one factored by this factor till it can be ascertained how many times it is common.

Find the greatest common divisor of 120, 168, 252, 432.

$$120=\underline{2}\times\cancel{5}\times\underline{2}\times\cancel{2}\times\underline{3}$$

$$\begin{array}{r|ccc} 2) & 168 & 252 & 432 \\ \hline 2) & 84 & 126 & 216 \\ \hline & 42 & 63 & 108 \end{array}$$

First factor 120 obtaining 2, 5, 2, 2, 3 as the prime factors. Then divide each of the other numbers by 2 twice in succession getting the quotients 42, 63 and 108. 2 will not divide these numbers and 5 will not, 3 will. Cross out 2 and 5 and underscore 3.

2×2×3=12, G. C. D.

Find the greatest common divisor of: (1) 81, 108, 135, 567. (2) 98, 112, 490, 336, 266. (3) 75, 45, 195, 435. (4) 44, 220, 264, 198, 681. (5) 117, 195, 273, 351.

Greatest common divisor is used in reducing fractions to their lowest terms. The result is the same whether the numerator and denominator are divided at once by their greatest common divisor or the common factors are removed in succession. Reduce $^{190}/_{209}$ to its lowest terms. 190=2×5×19. 2 and 5 are not common. Divide by 19 and the answer $^{10}/_{11}$ is obtained. $^{190}/_{209}=$ $^{10}/_{11}$.

Reduce the following fractions to their lowest terms:

1. $^{58}/_{145}$.
2. $^{66}/_{154}$.
3. $^{63}/_{105}$.
4. $^{161}/_{399}$.
5. $^{72}/_{324}$.

Common factors may be cast out in indicated division.

$$\frac{\overset{2}{\cancel{14}}\times\overset{\overset{2}{\cancel{6}}}{\cancel{18}}\times\overset{7}{\cancel{35}}}{\underset{3}{\cancel{21}}\times\underset{5}{\cancel{15}}}$$

$=7\times2\times2=28$ Ans.

LEAST COMMON MULTIPLE.

It is evident that a multiple of a number must contain all its prime factors, and that a common multiple of the two or more numbers must contain the prime factors of each and contain each prime factor as many times as it occurs in any one of the numbers.

To find the least common multiple of 24, 36, 60 and 72.

24=2×2×2×3
36=2×2×3×3
60=2×5×2×3
72=2×2×2×3×3

2×2×2×3×3×5=360 L. C. M.

The work of multiplying may be shortened by taking one of the numbers and multiplying it by the factors of the other numbers not found in it. Thus, in the problem solved above, take 72 which contains 2, 2, 2, 3, 3, and multiply it by 5. 72×5 =360 Ans.

Find the L. C. M. of 28, 42, 63.

28=2×2×7
42=2×3×7
63=3×3×7

28×3×3=28×9=252 Ans.

Find the L. C. M.:

1. 36, 54, 81.
2. 16, 24, 72.
3. 35, 42, 56.
4. 38, 57, 54.
5. 45, 60, 84, 126.

Sometimes the factors of each number may be seen by inspection. In such examples it is unnecessary to write out the factors.

Find the L. C. M. of 15, 21, 35,

42. It may be readily seen that the factors 2, 3, 5, 7 occur not more than once in the numbers. The L. C. M. is, therefore, $42 \times 5 = 210$ Ans.

Find the L. C. M. of the following without writing the factors of each if possible.

1. 18, 21, 24.
2. 22, 33, 55.
3. 35, 45, 55.
4. 28, 24, 42.
5. 6, 8, 10, 15, 25.

If one of the given numbers is a factor of another it may be omitted.

Find the L. C. M. of 14, 21, 35, 42. Omit 14 and 21 for each will divide 42.

$$35 = 5 \times 7$$
$$42 = 2 \times 3 \times 7$$
$$35 \times 2 \times 3 = 35 \times 6 = 210$$

Find the L. C. M.:

1. 2, 3, 4, 6, 8, 9.
2. 7, 14. 28, 42, 84.
3. 15, 30, 45, 75.
4. 8, 16, 24, 36, 48.
5. 22, 44, 66, 88, 99.

The least common multiple of small numbers may usually be found by inspection. This plan should be followed when possible.

Find the L. C. M.:

1. 3, 4, 8, 12.
2. 4, 8, 16, 20.
3. 6, 9, 12, 18.
4. 7, 14, 21.
5. 15, 25, 30.

Least Common Multiple is of use in reducing fractions to a common denominator as a preparation for addition or subtraction.

Reduce $\frac{7}{8}$, $\frac{5}{12}$, $\frac{11}{18}$, $\frac{2}{9}$, to their least common denominator.

The L. C. M. of 8, 9, 12, 18 is 72.

$$\frac{7}{8} = \frac{63}{72}$$
$$\frac{2}{9} = \frac{16}{72}$$
$$\frac{5}{12} = \frac{30}{72}$$
$$\frac{11}{18} = \frac{44}{72}$$

Reduce to least common denominator, finding the common denominator by inspection if possible.

1. $\frac{3}{4}$, $\frac{2}{5}$, $\frac{7}{10}$, $\frac{11}{20}$.
2. $\frac{5}{8}$, $\frac{7}{16}$, $\frac{11}{12}$, $\frac{3}{4}$, $\frac{13}{24}$.
3. $\frac{2}{3}$, $\frac{2}{5}$, $\frac{4}{15}$, $\frac{5}{6}$.
4. $\frac{4}{7}$, $\frac{1}{3}$, $\frac{5}{14}$, $\frac{11}{21}$.
5. $\frac{3}{5}$, $\frac{5}{7}$, $\frac{9}{14}$, $\frac{7}{10}$, $\frac{25}{35}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

A good form is helpful in addition and subtraction of fractions. The one here given is compact, businesslike, and free from objectionable features, and should be used exclusively after its significance is understood through the use of the full form.

Add: $4\frac{3}{7} + 7\frac{2}{5} + 9\frac{1}{2} + 17\frac{5}{14} + 11\frac{7}{10}$

	70
$4\frac{3}{7}$	30
$7\frac{2}{5}$	28
$9\frac{1}{2}$	35
$17\frac{5}{14}$	25
$11\frac{7}{10}$	49
$50\frac{27}{70}$	$\frac{167}{70}$

$$\frac{167}{70} = 2\frac{27}{70}$$

The final answer $50\frac{27}{70}$, is the last thing written. Any other work besides that here written should be done by inspection or as side work.

Add:

1. $7\frac{3}{4} + 9\frac{1}{3} + 4\frac{5}{6} + 11\frac{7}{12}$.
2. $27\frac{3}{5} + 95\frac{5}{8} + 47\frac{3}{4} + 5\frac{7}{10}$.
3. $\frac{2}{5} + 40\frac{1}{8} + 9\frac{4}{15} + 624\frac{3}{10} + 15\frac{7}{12}$.
4. $26\frac{5}{8} + 47\frac{4}{7} + 4\frac{15}{28} + \frac{9}{14}$.
5. $417\frac{7}{11} + 4\frac{9}{22} + 70\frac{8}{33} + 29\frac{5}{6}$.

When there are few fractions and the common denominator is small all the work may be done by inspection and only the fractions and the result written.

Add 36, $43\frac{1}{2}$, $18\frac{3}{4}$, and $6\frac{7}{8}$.

$$\begin{array}{r} 36 \\ 43\frac{1}{2} \\ 18\frac{3}{4} \\ 6\frac{7}{8} \\ \hline 105\frac{1}{8} \end{array}$$

Add:

1. $32\frac{3}{4} + 406\frac{1}{2} + 82\frac{3}{8} + 17\frac{1}{4}$.
2. $8\frac{5}{6} + 12\frac{1}{2} + 260\frac{1}{3} + 507$.
3. $43\frac{3}{5} + 18\frac{2}{3} + 16\frac{7}{15} + 200\frac{1}{5} + 60$.
4. $86\frac{1}{6} + 25\frac{3}{4} + 624\frac{2}{3} + 47\frac{5}{12}$.
5. $26\frac{1}{4} + 6\frac{2}{5} + 70\frac{7}{10} + 429\frac{1}{2} + \frac{4}{5}$.

The same form is used in subtracting a fraction.

Subtract $17\frac{4}{7}$ from $43\frac{3}{4}$.

$$\begin{array}{r|r} & 28 \\ \hline 43\frac{3}{4} & 21 \\ 17\frac{4}{7} & 16 \\ \hline 26\frac{5}{21} & \frac{5}{21} \end{array}$$

Subtract:

1. $\frac{5}{8} - \frac{2}{5}$.
2. $182\frac{13}{18} - 56\frac{5}{12}$.
3. $260\frac{7}{9} - 184$.
4. $235\frac{7}{15} - 68\frac{2}{9}$.
5. $902\frac{9}{14} - 364\frac{3}{8}$.

When the fractional part of the subtrahend is larger than the corresponding part of the minuend, a similar method is used.

Subtract $4\frac{5}{9}$ from $8\frac{1}{6}$.

$$\begin{array}{r|r} & 18 \\ \hline 8\ \frac{1}{6} & 3 \\ 4\ \frac{5}{9} & 10 \\ \hline 3\frac{11}{18} & \frac{11}{18} \end{array}$$

One is taken from the 8 and changed to 18th, giving $\frac{18}{18}$, the numerator being the same as the number above 3. 10 taken from 18 leaves 8 and 8 plus 3 equals 11.

Subtract

1. $72\frac{2}{9} - 38\frac{2}{3}$.
2. $174\frac{3}{8} - 89\frac{7}{12}$.
3. $500 - 43\frac{4}{7}$.
4. $294\frac{11}{24} - 49\frac{3}{4}$.
5. $14\frac{3}{7} - \frac{4}{5}$.

Give the results in the following by inspection:

1. $\frac{3}{4} - \frac{3}{8}$.
2. $14\frac{5}{6} - 9\frac{1}{2}$.
3. $26\frac{1}{2} - 17\frac{2}{5}$.
4. $13\frac{3}{4} - 6\frac{5}{12}$.
5. $20\frac{6}{7} - 8\frac{2}{3}$.
6. $48 - 37\frac{2}{5}$.
7. $27\frac{1}{2} - \frac{7}{8}$.
8. $65\frac{4}{9} - 57\frac{2}{5}$.
9. $73\frac{3}{11} - 64\frac{1}{2}$.
10. $18\frac{3}{8} - 6\frac{7}{12}$.

To Multiply a Fraction by a Whole Number.

In multiplication and division of fractions it is best to memorize a clear concise rule.

To multiply a fraction by a whole number, multiply the numerator by the whole number and place the result over the denominator.

Multiply $\frac{6}{7}$ by 5.

$$5 \times \frac{6}{7} = \frac{30}{7} = 4\frac{2}{7}.$$

Multiply:

1. $\frac{7}{8}$ by 9.
2. $\frac{15}{16}$ by 12.
3. $1\frac{3}{4}$ by 6.
4. $2\frac{2}{5}$ by 14.
5. $\frac{15}{16}$ by 18.
6. $\frac{11}{21}$ by 15.

The work may be indicated and the process often shortened by cancellation.

Multiply $2\frac{3}{8}$ by 28.

$$28 \times 2\frac{3}{8} = \frac{\overset{7}{\cancel{28}} \times 19}{\underset{2}{\cancel{8}}} = \frac{133}{2} = 66\frac{1}{2}.$$

Multiply:

1. $\frac{7}{12}$ by 18.
2. $\frac{5}{14}$ by 35.
3. $2\frac{7}{9}$ by 15.
4. $5\frac{3}{28}$ by 32.
5. $7\frac{2}{3}$ by 8.
6. $17\frac{4}{7}$ by 9.

If the integral part of a mixed number is large it is not best to reduce to an improper fraction before multiplying. Where the multiplier consists of but one figure the reduction and carrying can be done at once.

Multiply $27\frac{3}{4}$ by 7.

$$\begin{array}{r} 27\frac{3}{4} \\ 7 \\ \hline 194\frac{1}{4} \end{array}$$

Multiply:

1. $126\frac{5}{7}$ by 9.
2. $237\frac{3}{8}$ by 6.
3. $46\frac{5}{12}$ by 7.
4. $2649\frac{4}{11}$ by 8.
5. $4285\frac{7}{12}$ by 5.

If the multiplier consists of two or more digits it is best to multiply the fraction and whole number separately.

Multiply $67\frac{5}{8}$ by 29.

$$\begin{array}{r} 67\frac{5}{8} \\ 29 \\ \hline 18\frac{1}{8} \\ 603\phantom{\frac{1}{8}} \\ 134\phantom{0\frac{1}{8}} \\ \hline 1961\frac{1}{8} \end{array}$$

Multiply:
1. $358\frac{5}{9}$ by 28.
2. $75\frac{7}{12}$ by 32.
3. $2674\frac{8}{15}$ by 25.
4. $1960\frac{5}{11}$ by 24.
5. $3602\frac{7}{16}$ by 48.

To Divide a Fraction by a Whole Number.

To divide a fraction by a whole number divide the numerator or multiply the denominator by the whole number.

Divide $4\frac{4}{5}$ by 6.

$$4\frac{4}{5} \div 6 = \frac{24}{5} \div 6 = \frac{4}{5}$$

Divide $\frac{1}{4}$ by 3

If each fourth is divided by 3 it will be seen that the whole is divided into twelve parts. Hence

$$\frac{1}{4} \div 3 = \frac{1}{12}.$$

Divide:
1. $\frac{28}{9}$ by 4.
2. $\frac{35}{8}$ by 7.
3. $8\frac{4}{7}$ by 12.
4. $8\frac{3}{4}$ by 5.
5. $12\frac{4}{9}$ by 7.
6. $67\frac{1}{11}$ by 9.

Divide $1\frac{8}{9}$ by 3.

$$1\frac{8}{9} \div 3 = \frac{17}{9} \div 3 = \frac{17}{27}.$$

Divide:
1. $\frac{9}{10}$ by 4.
2. $2\frac{3}{7}$ by 5
3. $4\frac{5}{9}$ by 6.
4. $12\frac{3}{4}$ by 8.
5. $6\frac{1}{4}$ by 5.

The operation may be indicated and the work often shortened by cancellation.

Divide $6\frac{3}{4}$ by 12.

$$6\frac{3}{4} \div 12 = \frac{27}{4} \div 12 = \frac{\overset{9}{\cancel{27}}}{4\times\underset{4}{\cancel{12}}} = \frac{9}{16}.$$

Divide:
1. $\frac{14}{15}$ by 21.
2. $7\frac{2}{9}$ by 10.
3. $16\frac{2}{3}$ by 20.
4. $7\frac{3}{8}$ by 9.
5. $\frac{44}{65}$ by 33.
6. $7\frac{1}{11}$ by 39.

When the integral part of a mixed number is larger than the divisor it is not best to reduce to an improper fraction at first. If the divisor is small use short division.

Divide $187\frac{3}{7}$ by 8.

$$\begin{array}{r} 23\frac{3}{7} \\ \hline 8)187\frac{3}{7} \end{array}$$

Side work:

$$3\frac{3}{7} \div 8 = \frac{24}{7} \div 8 = \frac{3}{7}$$

Divide:
1. $43\frac{3}{4}$ by 7.
2. $29\frac{5}{8}$ by 9.
3. $265\frac{4}{9}$ by 8.
4. $1982\frac{5}{6}$ by 8.
5. $6371\frac{7}{8}$ by 5.
6. $7681\frac{3}{8}$ by 6.

If the divisor is large use long division.

Divide $6744\frac{5}{9}$ by 17.

$$\begin{array}{l} \quad\ \ 396\frac{122}{153} \\ \hline 17)6745\ \ \frac{5}{9} \\ \quad\ \ 51 \\ \quad\ \ \overline{164} \\ \quad\ \ 153 \\ \quad\ \ \overline{\ \ 115} \\ \quad\ \ \ \ 102 \\ \quad\ \ \overline{\ \ \ \ 13\ \frac{5}{9}} \end{array}$$

Side work:

$$13\frac{5}{9} = \frac{122}{9}$$

$$\frac{122}{9} \div 17 = \frac{122}{153}$$

Divide:
1. $86\frac{4}{7}$ by 14.
2. $2763\frac{3}{8}$ by 23.
3. $1679\frac{5}{12}$ by 18.
4. $3462\frac{5}{6}$ by 26.
5. $7632\frac{3}{8}$ by 15.
6. $87652\frac{7}{11}$ by 21.

To Multiply a Whole Number by a Fraction.

Since the product of two factors is the same no matter which is used as the multiplier the rule for multiplying a fraction by a whole number may be used. When the whole number is large it is frequently desirable to use the fraction as the multiplier.

To multiply a whole number by a fraction multiply the number by the numerator and divide the product by the denominator:

Multiply 47 by $\frac{3}{8}$.

$\frac{3}{8}\times 47 = \frac{141}{8} = 17\frac{5}{8}$ Ans.

Multiply:

1. $65 \times \frac{5}{9}$.
2. $87 \times \frac{6}{7}$.
3. $432 \times \frac{5}{9}$.
4. $76 \times \frac{8}{11}$.
5. $43 \times 1\frac{3}{5}$.
6. $216 \times 2\frac{3}{4}$.

It is not necessary and frequently not desirable to reduce a mixed number to an improper fraction before multiplying. It is best in such cases to multiply by the fractional part first.

Multiply 89 by $27\frac{3}{8}$.

$$\begin{array}{r} 89 \\ 27\frac{3}{8} \\ \hline 8)267 \\ \hline 33\frac{3}{8} \\ 623\phantom{\frac{3}{8}} \\ 178\phantom{0\frac{3}{8}} \\ \hline 2436\frac{3}{8} \end{array}$$

Multiply:

1. 27 by $\frac{5}{12}$.
2. 44 by $1\frac{3}{8}$.
3. 45 by $\frac{8}{15}$.
4. 88 by $\frac{7}{8}$.
5. 496 by $29\frac{3}{4}$.
6. 57851 by $43\frac{5}{9}$.

To multiply a fraction by a fraction **multiply the numerators for the numerator and the denominators for the denominator of the product.**

Multiplying a fraction by a fraction is best explained by the use of a diagram. Multiply $\frac{1}{5}$ by $\frac{1}{2}$.

$\frac{1}{2}$ of $\frac{1}{5} = \frac{1}{10}$, hence
$\frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$.
Similarly $\frac{1}{2} \times \frac{3}{5} = \frac{3}{10}$.

In like manner it may be shown that $\frac{2}{5} \times \frac{3}{7} = \frac{6}{35}$; $\frac{3}{4} \times \frac{5}{8} = \frac{15}{32}$; etc. Have the pupils find the products by using the diagram. By comparing the results with the factors, the rule may be deduced.

The operation should be indicated and cancellation used when possible. As a rule mixed numbers should be reduced to improper fractions before multiplying.

Multiply:

1. $\frac{7}{8}$ by $\frac{9}{14}$.
2. $\frac{12}{25}$ by $\frac{15}{28}$.
3. $2\frac{7}{9}$ by $3\frac{9}{20}$.
4. $\frac{8}{11} \times \frac{55}{84} \times \frac{15}{26}$.
5. $2\frac{5}{8} \times 4\frac{2}{7} \times 7\frac{1}{5}$.
6. $4\frac{4}{11} \times 5\frac{5}{9} \times 7\frac{7}{10} \times 3\frac{3}{8}$.

A compound fraction is reduced to a simple fraction by multiplication. Reduce $\frac{5}{9}$ of $\frac{7}{12}$ of $6\frac{3}{4}$ to a simple fraction:

$$\frac{5}{9} \text{ of } \frac{7}{12} \text{ of } 6\frac{3}{4} = \frac{5}{\cancel{9}} \times \frac{7}{\underset{4}{\cancel{12}}} \times \frac{\overset{\cancel{3}}{\cancel{27}}}{4}$$

$$= \frac{35}{16} = 2\frac{3}{16}$$

Simplify:

1. $\frac{6}{7}$ of $4\frac{3}{8}$.
2. $\frac{5}{9}$ of $6\frac{3}{7} + \frac{8}{11}$ of $2\frac{5}{14}$.
3. $\frac{5}{8}$ of $2\frac{3}{4}$ of $\frac{7}{9}$.
4. $\frac{6}{7}$ of $2\frac{4}{5}$ of $1\frac{7}{15} - \frac{3}{4}$ of $2\frac{2}{5}$.
5. $\frac{21}{25}$ of $\frac{15}{28} + \frac{7}{11}$ of $6\frac{3}{5} - 2\frac{5}{8}$.

When the integral part of a mixed number is large, it is best not to redure to an improper fraction. In such cases multiply thru by the numerator of the multiplier and divide the result by the denominator.

Multiply $286\frac{5}{7}$ by $\frac{8}{9}$. (Side work)

$$\begin{array}{rl} 286\frac{5}{7} & \\ \frac{8}{9} & \\ \hline 9)2293\frac{5}{7} & \text{multiplying by 8.} \\ \hline 254\frac{6}{7} & \text{dividing by 9.} \\ \hline \end{array}$$

$7\frac{5}{7} = \frac{54}{7}$
$\frac{54}{7} \div 9 = \frac{6}{7}$

Multiply:

1. $796\frac{5}{8}$ by $\frac{7}{9}$.
2. $384\frac{3}{11}$ by $\frac{6}{7}$.
3. $49\frac{4}{13}$ by $\frac{8}{11}$.
4. $3492\frac{5}{9}$ by $\frac{3}{8}$.
5. $4628\frac{7}{15}$ by $\frac{4}{9}$.
6. $937\frac{11}{14}$ by $\frac{8}{11}$.

If the multiplier is a mixed number, multiply through by the fraction as shown above and then by the integral part.

Multiply $679\frac{4}{7}$ by $8\frac{5}{9}$. (Side work)

$$
\begin{array}{ll}
679\ \frac{4}{7} & \\
 & 4\frac{6}{7} = \frac{34}{7} \\
8\ \frac{5}{9} & \frac{34}{7} \div 9 = \frac{34}{63} \\
\hline
9)3397\ \frac{6}{7} & |\ \text{Multiplying by } 5 \\
\hline
377\frac{34}{63} & |\ 34 \\
5436\ \frac{4}{7} & |\ 36 \\
\hline
5814\ \frac{1}{9} & |\ \frac{70}{63} = 1\frac{1}{9}
\end{array}
$$

If the integral part of the multiplier is large multiply the fraction by it and write the result as a part of the product, then muliply the integral parts together.

Multiply $4758\frac{7}{11}$ by $37\frac{3}{4}$.

$$
\begin{array}{ll}
4758\ \frac{7}{11} & \\
37\frac{3}{4} & \\
\hline
4)\ 14275\frac{10}{11} & \text{Multiplying by } 3 \\
\hline
3568\frac{43}{44} & \text{dividing by } 4 \\
23\ \frac{6}{11} & \\
33306 & \\
14274 & \\
\hline
179638\frac{23}{44} &
\end{array}
$$

(Side work)

$$
\begin{aligned}
3\tfrac{10}{11} &= \tfrac{43}{11} \\
\tfrac{43}{11} \div 4 &= \tfrac{43}{44} \\
37 \times \tfrac{7}{11} &= \tfrac{259}{11} \\
&= 23\tfrac{6}{11}
\end{aligned}
$$

Multiply:

1. $416\frac{2}{3}$ by $7\frac{2}{5}$.
2. $3652\frac{3}{4}$ by $9\frac{4}{7}$.
3. $6942\frac{5}{6}$ by $8\frac{3}{8}$.
4. $5873\frac{4}{11}$ by $25\frac{1}{2}$.
5. $3254\frac{6}{13}$ by $43\frac{7}{12}$.
6. $2735\frac{3}{5}$ by $369\frac{2}{9}$.

To Divide a Whole Number by a Fraction.

In order to find how many times 4 ft. is contained in 8 yards it is necessary to change both numbers to the same kind.

8 yds. = 24 ft.
24 ft. ÷ 4 ft. = 6 Ans.

Divide 10 ft. by 15 in.:

10ft. = 120 in.
120 in. ÷ 15 in = 8 Ans.

In like manner when a whole number is to be divided by a fraction the whole number should be reduced to a fraction having the same denominator as the divisor. Hence the rule:

To divide a whole number by a fraction multiply the whole number by the denominator and divide the result by the numerator, or multiply the whole number by the divisor inverted.

Divide 28 by $\frac{5}{9}$.

$$28 \div \tfrac{5}{9} = \frac{28 \times 9}{5} = \tfrac{252}{5} = 50\tfrac{2}{5}.$$

Divide 56 by $5\frac{1}{4}$:

$$56 \div 5\tfrac{1}{4} = \overset{8}{\cancel{56}} \times \frac{4}{\underset{3}{\cancel{21}}} = \tfrac{32}{3} = 10\tfrac{2}{3}$$

Divide:

1. 37 by $\frac{4}{7}$.
2. 42 by $2\frac{2}{5}$.
3. 78 by $7\frac{3}{7}$.
4. 5 by $7\frac{1}{2}$.
5. 16 by $5\frac{1}{3}$.
6. 65 by $\frac{4}{7}$ of $6\frac{1}{2}$.

If the whole number is large and the denominator of the fraction is small, write the divisor to the left:

Divide 4763 by $5\frac{2}{3}$.

Change both to thirds:

$$
\begin{array}{ll}
5\frac{2}{3})\ 4763 & \\
17\ \ )14289 &
\end{array}
\qquad
\begin{array}{r}
840\frac{9}{17} \\
17)\overline{14289} \\
136 \\
\hline
68 \\
68 \\
\hline
9
\end{array}
$$

Divide:

1. 2769 by $2\frac{1}{4}$.
2. 11483 by $1\frac{3}{5}$.
3. 7925 by $3\frac{2}{7}$.
4. 37261 by $1\frac{5}{6}$.
5. 17984 by $2\frac{4}{9}$.
6. 21358 by $7\frac{3}{4}$.

In the exercises in division of fractions so far solved the entire quotient including both fractional and integral parts has been obtained. It is sometimes desirable to have only the integral part of the quotient and to know what the remainder is. The remainder is the undivided part of the dividend, hence is of the same

kind. For example, if dresses each requiring 9 yards are made from a piece of cloth containing 67 yards, 7 dresses can be made and there will be a remainder of 4 yards.

The remainder may be found in either of two ways: 1st, By multiplying the divisor by the integral quotient and subtracting the product from the dividend; 2nd, By taking the remainder after the last division, noting what it is.

In the last model given above, the integral quotient is 840 and the remainder is $\frac{9}{3}$ or 3, since both dividend and divisor were changed to thirds. It may be obtained by the first way given above as follows:

Dividend	4763
$840 \times 5\frac{2}{3} =$	4760
Rem.	3

Find the integral quotients and remainders in the problems in last two exercises given above.

To Divide a Fraction by a Fraction.

If the fractions have a common denominator, divide the numerator of the dividend by the numerator of the divisor.

Divide $7\frac{1}{5}$ by $\frac{4}{5}$.

$7\frac{1}{5} \div \frac{4}{5} = \frac{36}{5} \div \frac{4}{5} = 9$ Ans.

Divide:

1. $\frac{24}{35}$ by $\frac{3}{35}$.
2. $6\frac{3}{4}$ by $\frac{3}{4}$.
3. $18\frac{1}{3}$ by $3\frac{2}{3}$.
4. $\frac{8}{9}$ by $\frac{5}{9}$.
5. $\frac{3}{7}$ by $4\frac{2}{7}$.
6. $7\frac{2}{9}$ by $2\frac{1}{9}$.

Give the entire quotient in each example here given; also the integral quotient and the remainder in all except example 5.

If the common denominator is small, reduce the fractions to a common denominator and divide as above.

Divide $7\frac{1}{2}$ by $1\frac{1}{4}$.

$7\frac{1}{2} \div 1\frac{1}{4} = \frac{30}{4} \div \frac{5}{4} = 6$ Ans.

Divide:

1. $8\frac{2}{3}$ by $2\frac{1}{6}$.
2. $15\frac{3}{4}$ by $\frac{7}{12}$.
3. $14\frac{2}{5}$ by $\frac{8}{15}$.
4. $\frac{11}{8}$ by $\frac{1}{4}$.
5. $\frac{5}{6}$ by $\frac{3}{4}$.
6. $6\frac{2}{3}$ by $8\frac{1}{2}$.

Give the entire quotient in each of the examples here given, also the integral quotient and remainder in all except example 6.

If the dividend is a mixed number with large integral parts and the common denominator is small, write the divisor to the left:

Divide $687\frac{1}{2}$ by $2\frac{2}{3}$. Reduce both to sixths:

```
          257 13/16
       ------------
  2 2/3) 687 1/2
       ------------
 16     )4125
         32
         --
          92
          80
          --
          125
          112
          ---
           13
```

Ans. $257\frac{13}{16}$, or 257 Quo. and $2\frac{1}{6}$ Rem.

Divide:

1. $2647\frac{2}{3}$ by $1\frac{3}{4}$.
2. $675845\frac{5}{8}$ by $3\frac{1}{2}$.
3. $473\frac{4}{5}$ by $7\frac{1}{2}$.
4. $358\frac{3}{8}$ by $2\frac{3}{4}$.
5. $627\frac{5}{6}$ by $4\frac{1}{3}$.
6. $3721\frac{3}{4}$ by $7\frac{3}{5}$.

A fraction may be divided by a fraction by applying the following rule:

To divide a fraction by a fraction, invert the divisor and proceed as in multiplication.

Divide $4\frac{2}{7}$ by $\frac{5}{8}$:

$4\frac{2}{7} \div \frac{5}{8} = \frac{30}{7} \times \frac{8}{5} = \frac{48}{7} = 6\frac{6}{7}$.

When problems are solved in this manner, cancellation may be used and the entire quotient only is obtained. If the integral quotient and the remainder are desired the remainder may be found by multiplying and subtracting as previously given.

		28
Dividend	$4\frac{2}{7}$	8
$6 \times \frac{5}{8} =$	$3\frac{3}{4}$	21
	$\frac{15}{28}$	$\frac{15}{28}$

Hence the integral quotient is 6 and the remainder $\frac{15}{28}$.

Divide:

1. $\frac{8}{9}$ by $\frac{4}{7}$.
2. $3\frac{3}{7}$ by $5\frac{3}{14}$.
3. $\frac{4}{9}$ of $6\frac{3}{7}$ by $7\frac{1}{2}$.
4. $\frac{5}{8}$ of $12\frac{4}{7}$ by $\frac{3}{4}$ of $2\frac{5}{14}$.
5. $7\frac{1}{3}$ by $\frac{4}{9}$ of $6\frac{3}{5}$.
6. $\frac{2}{3}$ of $\frac{6}{7}$ of $3\frac{3}{8}$ by $\frac{5}{7}$ of $4\frac{1}{2}$.

A complex fraction is merely an indicated divisor in which the numerator is the dividend and the denominator the divisor. It can be simplified by performing the indicated operations.

Simplify $\dfrac{6\frac{3}{7}}{8\frac{1}{3}}$

$$\frac{6\frac{3}{7}}{8\frac{1}{3}} = 6\tfrac{3}{7} \div 8\tfrac{1}{3} = \frac{\overset{9}{\cancel{45}}}{7} \times \frac{3}{\underset{5}{\cancel{25}}} = \frac{27}{35}$$

Simply:

1. $\dfrac{\frac{5}{8}}{3\frac{3}{4}}$
2. $\dfrac{\frac{6}{7} \text{ of } 4\frac{2}{5}}{7\frac{7}{15}}$
3. $\dfrac{7\frac{1}{2} + 4\frac{3}{4}}{5\frac{1}{4}}$
4. $\dfrac{12\frac{5}{6} - 4\frac{2}{3}}{12\frac{5}{6} + 4\frac{2}{3}}$
5. $\dfrac{\frac{3}{4} \text{ of } 8\frac{4}{5} - 1\frac{7}{10}}{14 - \frac{2}{3} \text{ of } 5\frac{1}{4}}$
6. $\dfrac{\frac{5}{9} \div \frac{7}{8}}{\frac{3}{5} \times 1\frac{3}{4}}$

THE USE OF SIGNS.

1. If only the signs $+$ and $-$ occur in an expression the operations are to be performed in order from left to right.

$$6 + 7 - 3 - 2 + 4 = 12.$$

2. If only the signs $\times$ and $\div$ occur in an expression the operations are to be performed in order from left to right.

$$48 \div 6 \times 2 = 16$$
$$48 \div 2 \times 6 = 144$$
$$48 \div 6 \div 2 = 4$$

3. If the signs $+$, $-$, $\times$, and $\div$ occur in an expression the multiplications and divisions are to be performed first, and then the additions and subtractions.

$$12+3\times2-9\div3=12+6-3=15$$

Find the value:

1. $4\times7-8\div2+6\times3$.
2. $27-12\div3+2$.
3. $18+9\times4\div6-4$.
4. $16\times3+24\div8\times2-18\div6$.
5. $7\times5+10-4\times5+3$.

Simplify:

1 $\dfrac{1\frac{1}{4}\times1\frac{2}{7}+\frac{1}{3} \text{ of } 2\frac{1}{4}-\frac{13}{28}\times2.}{\frac{13}{28} \text{ of } 2+\frac{1}{3} \text{ of } 2\frac{1}{4}-1\frac{1}{4} \text{ of } 1\frac{2}{7}}$

Ans. 20.

2. $2\frac{1}{4}\times\dfrac{10\frac{3}{4}-4\frac{11}{12}}{6\frac{3}{16}+7\frac{2}{3}}\times\dfrac{3\frac{5}{11}}{1\frac{2}{5}\times9\frac{1}{11}}$

Ans. $\frac{9}{35}$

3. $\dfrac{\frac{1}{6} \text{ of } 1\frac{13}{16}+1\frac{1}{6} \text{ of } 6\frac{1}{4}-1\frac{1}{3} \text{ of } 5\frac{4}{9}}{\frac{1}{6} \text{ of } 2\frac{5}{6} \text{ of } 5\frac{2}{3}}$

Ans. $\frac{1}{8}$

4. $\dfrac{3\frac{7}{9}\times1\frac{1}{17}+4\frac{1}{12}-3\frac{9}{16}}{5\frac{1}{9}-7\frac{7}{8}\div28\frac{7}{20}+\frac{1}{3}}$ Ans. $\frac{7}{8}$

5. $\dfrac{6\frac{3}{4}+5\frac{1}{2}\times3\frac{1}{7}-7\frac{1}{4}}{3\frac{1}{5}+2\frac{1}{2}-4\frac{1}{10}}$ Ans. $10\frac{55}{112}$

6. $\dfrac{5\frac{5}{8}\div\frac{2}{3}}{1\frac{1}{5} \text{ of } \frac{5}{9}\div10\frac{1}{3}}\times\frac{2}{5}$ of $\dfrac{1\frac{1}{2} \text{ of } 4\frac{1}{9}}{13\frac{7}{8} \text{ of } 5\frac{1}{3}}$ Ans. $4\frac{23}{64}$

7. $\dfrac{1\frac{4}{5}-\frac{2}{3} \text{ of } 1\frac{1}{4} \text{ of } 1\frac{1}{5}}{1\frac{1}{20}-\frac{2}{9} \text{ of } 2\frac{3}{8}}$ Ans. $1\frac{25}{47}$

8. $\dfrac{\frac{3}{4} \text{ of } \frac{5}{6}\div1\frac{2}{3}}{2\frac{1}{4}\times\frac{5}{18} \text{ of } \frac{3}{5}}\times1\frac{1}{4}\times1\frac{1}{5}$.

Ans. $1\frac{1}{2}$

9. $\dfrac{\frac{3}{4}\times\frac{2}{3}\div2+\frac{4}{5}-\frac{3}{10}\div\frac{1}{20}\times\frac{1}{6}}{\frac{\frac{3}{4}}{\frac{2}{3}}\times\frac{\frac{2}{3}}{\frac{3}{4}}\div20}$ Ans. 1

Simplify the following:

1. $\frac{13}{8}$ of $\frac{10}{17}-\dfrac{5\frac{2}{3}-3\frac{1}{2}}{4\frac{1}{4}+2\frac{5}{6}}+\frac{9}{10}$ of $3\frac{5}{6}$. Ans. $4\frac{1}{10}$

2. $\dfrac{1\frac{7}{10}+1\frac{2}{13}+3\frac{9}{16}\times1\frac{1}{19}}{3\frac{5}{6}\div2\frac{5}{9}-4\frac{4}{9}\div10\frac{5}{6}}$ Ans. $6\frac{3}{50}$

3. $\dfrac{3\frac{1}{2}-2\frac{2}{3}}{4\frac{1}{4}-3\frac{1}{3}}+\frac{4}{9}$ of $3\frac{3}{5}-1\frac{7}{20}\div\frac{18}{25}-\frac{21}{40}$ Ans. $\frac{6}{55}$

4. $\dfrac{4\frac{7}{12}\times1\frac{11}{40}-3\frac{9}{14}\div5\frac{1}{7}}{(7\frac{2}{9}+6\frac{7}{8})\div(8\frac{7}{12}-5\frac{5}{18})}$ Ans. $1\frac{49}{240}$

5. $\dfrac{\frac{22}{21}\times\frac{123}{46}-\frac{22}{15}\times\frac{27}{14}}{\frac{25}{18}\div\frac{20}{63}-\frac{15}{16}\div\frac{9}{20}}$ Ans. $2\frac{16}{25}$

6. $\dfrac{\frac{18}{85}\text{ of }18\frac{8}{9}+8\frac{5}{6}-4\frac{7}{16}}{4\frac{1}{3}-3\frac{15}{16}\div14\frac{7}{40}+1\frac{1}{9}}$ Ans. $1\frac{5}{8}$

7. $\dfrac{\frac{3}{8}\text{ of }22\frac{1}{2}}{1\frac{5}{6}\div2\frac{1}{5}-\frac{1}{3}}\times\dfrac{8\frac{2}{9}\div\frac{2}{3}}{69\frac{3}{8}\div\frac{3}{16}}-\frac{23}{54}$. Ans. $\frac{59}{432}$

8. $\dfrac{5\frac{1}{3}\times2\frac{7}{12}}{8\frac{6}{7}\times1\frac{1}{2}}+\dfrac{1\frac{5}{12}}{\frac{3}{5}\text{ of }1\frac{8}{9}}-2\frac{1}{12}\div2\frac{11}{17}$ Ans. $1\frac{1}{2}$

9. $\dfrac{\frac{44}{63}\div\frac{40}{369}+5\frac{13}{15}\div1\frac{1}{27}}{6\frac{17}{18}\times3\frac{3}{20}-4\frac{11}{16}\times2\frac{2}{9}}$. Ans. $1\frac{7}{125}$

DECIMALS.

In decimal fractions the denominator is 10, 100, 1000, or, in general, 1 with one or more ciphers annexed. It is not written, but is indicated by a dot called a decimal point placed in the numerator.

The number of places to the right of the decimal point in the numerator is the same as the number of ciphers after the one in the denominator. Thus, in .4, .37, 6.145 the denominators are 10, 100 and 1000, respectively.

If the decimal has no other fraction attached it is a pure or a mixed decimal.

In a pure decimal the numerator is less than the denominator. Hence there are no significant figures to the left of the decimal point. .17, .809 and .0653 are pure decimals. These correspond to proper fractions.

In a mixed decimal the numerator equals or exceeds the denominator. Hence there must be some significant figure or figures to the left of the decimal point. 3.7, 60.08, 1.2374 are mixed decimals. They correspond to mixed numbers and improper fractions.

In reading pure decimals, read the numerator and then give the name of the denominator. Thus, .26 is read twenty-six hundredths, .0026 is read twenty-six ten-thousands, .00326 is read three hundred twenty-six hundred-thousands.

Read: .704, .0470, .600, .00006, .0400, .410, .000910, .0000900, .0572, .5072.

Mixed decimals may be read in two ways: as a mixed number, and as an improper fraction. When read as a mixed number **and** is used at the decimal point only, for this separates the integral from the fractional part. Thus, 4.9, 27.432, 600.028 are read four and nine-tenths, twenty-seven and four hundred thirty-two thousandths, six hundred and twenty-eight thousandths, respectively. In reading a mixed decimal as an improper fraction, read the entire numerator as it would be read if there were no decimal point and then give the name of the denominator. Thus, 6.7, 43.007, 75.0800, 9.60 are read 67 tenths, 43007 thousands, 750800 ten-thousandths, and 960 hundredths, respectively.

Read the following, first as mixed numbers and then as improper fractions:

60.009	9.0009
307.0705	150.7209
9.43075	61.4
600.7	100.008
600.007	5.9

728.46
72800.46
15.09
1500.09
700.008

Write the following:

1. 4 hundred-thousandths, 400 thousandths.

2. 3000 ten-millionths, 3010 millionths.

3. 805 tenths, 800 and 5 tenths.

4. 3076 hundredths, 3000 and 76 hundredths.

5. 500 ten-thousandths, 510 thousandths, and 500 and 10 thousandths.

Annexing a cipher to a decimal multiplies both numerator and denominator by ten and therefore does not change its value. Thus, 4.6=4.60=4.600=4.6000, etc. A whole number may be written as a decimal by placing a decimal point to its right and annexing one or more ciphers. Thus, 43 = 43.0 = 43.00 =43.000=43.0000, etc. Read both as a mixed decimal and as an improper fraction. Reduce 21 to tenths, to thousandths, to millionths.

Dropping a cipher at the right of a decimal divides both numerator and denominator by ten, hence does not change its value. Thus, 2.4000=2.400=2.40=2.4. Read each.

Moving the decimal point one place to the right divides the denominator by ten without changing the numerator, hence multiplies the fraction by 10. Name the numerator and the denominator in each of the following: .000726, .00726, .0726, .726, 7.26, 72.6, 726. Read each.

Moving the decimal point one place to the left multiplies the denominator by 10 without changing the numerator, hence it divides the fraction by 10. Name the numerator and the denominator in each of the following: 38, 3.8, .38, .038, .0038, .00038. Read each.

Reduce a Decimal to a Common Fraction.

A pure decimal may be changed to a common fraction by writing the denominator under the numerator, omitting the decimal point. The fraction may then be reduced to its lowest terms. Thus, $.072=\frac{72}{1000}=\frac{9}{125}$.

Reduce to common fractions and to their lowest terms:

.45	.144	.625
.045	.00144	.6205
.405	.10044	.00625

Mixed decimals may be treated either as mixed numbers or as improper fractions. Thus, $16.25=16\frac{25}{100}=16\frac{1}{4}$ or $16.25=\frac{1625}{100}=\frac{65}{4}=16\frac{1}{4}$.

Reduce the following both ways:

6.56	43.75	341.8
9.006	12.15	3.418
90.06	1.215	2.760

ADDITION AND SUBTRACTION OF DECIMALS.

Read the following: .3, .03, .003, .0003. It will be seen from the above that a figure in the first place to the right of the decimal point represents tenths; in the second, hundredths; in the third, thousandths; and so on. For this reason the orders to the right of the decimal point are called tenths, hundredths, thousandths, ten-thousandths, etc. Name the order represented by each figure in the following: 4063.259187. Note that in reading a decimal the name of the right hand order is given.

Note that 1=1.0, .1=.10, .01=.010, .001=.0010, and in general that ten units of any order makes one of the next higher. Hence in addition and subtraction of decimals, carrying is done the same as in whole numbers. Care must be taken that units of the same order stand under each other and are added.

Add:

1.
```
.247
.6053
.0958
.86
.09016
```

2.
```
 86.094
 32.7005
603.00951
720.900084
   .78
```

3. 8200 + 2.834 + 97.6005 + 306.942.

Subtract:

4.
```
 .784
—.329
```

5. 64.3
—28.56

6. 970.
—7.083

7. 23.6—2.36.
8. 7.4—.0659
9. .7—.0849.
10. 543.27—57.904.

MULTIPLICATION AND DIVISION OF DECIMALS.

The principles that apply in common fractions apply also in decimals, and the same rules for multiplication and division might be used. It is better, however, to use special rules in handling decimals.

To Multiply a Decimal by a Whole Number.

In common fractions the numerator is multiplied and the denominator is not changed. Likewise in decimals the number written, the numerator, is multiplied by the whole number and the same number of places pointed off.

Multiply 29.67 by 34.

```
  29.67
     34
 ------
  11868
  8901
 ------
1008.78
```

Multiply:
1. .084 by 149.
2. 63.828 by 365.
3. 920.05 by 76.
4. .475 by 357.
5. .625 by 96.

To Multiply a Decimal by One with one or more ciphers annexed.

Multiply .4768 by 1000.

```
.4768
 1000
-----
476.800   or   476.8
```

RULE—To multiply a decimal by one with ciphers annexed, move the decimal point as many places to the right as there are ciphers annexed to one.

Note that this is merely a kind of cancellation. Write the result at once. Thus, $4.672 \times 100 = 467.2$.

Multiply:
1. 67.853 by 100.
2. 6.3009 by 1000.
3. 7.48 by 10000.
4. .76 by 100.
5. .706 by 100000.

To Multiply a Decimal by Any Number with Ciphers Annexed.

Multiply .0674 by 4700.

```
 .0674
   4700
 ------
 471800
2696
 ------
316.7800   or   316.78.
```

RULE: Multiply as in whole numbers and move the decimal point as many places to the right as there are ciphers annexed in the multiplier.

When the multiplier is not larger than 12 with ciphers annexed, write the result at once. Thus, $4.7954 \times 7000 = 33567.8$.

Multiply:
1. 30.795 by 600.
2. .07532 by 9000.
3. 48.73 by 80000.
4. 2.876 by 4300.
5. 672.9 by 128000.
6. .00924 by 86000.

To Divide a Decimal by a Whole Number.

One way of dividing a fraction by a whole number is to divide the numerator by the whole number without changing the denominator. Applying this to decimals would give the rule here stated.

RULE: To divide a decimal by a whole number divide as in whole numbers and point off as many decimal places in the quotient as there are decimal places that have been used in the dividend.

The decimal point should be placed in the quotient as soon as it has been reached in the dividend. If the quotient is placed in its proper position over the dividend, the decimal point of the quotient will come directly over the decimal point of the dividend.

Note that there must be a figure in the quotient for every figure which has been used to the right of the decimal point in the dividend.

Divide 2937.97 by 47.

```
      62.51  Ans.
   --------
47)2937.97
   282
   ---
    117
     94
    ---
     239
     235
     ---
       47
       47
       --
```

Divide 4.6292 by 284.

```
        .0163  Ans.
    ------
284)4.6292
    2 84
    ----
    1 789
    1 704
    -----
      852
      852
      ---
```

Divide:

1. 95.88 by 94.
2. 3130.48 by 872.
3. .0201474 by 54.
4. 19.7635 by 841.
5. 3123.6 by 685.
6. 332.45 by 488.

The divisor is not always exactly contained in the dividend. There may be a remainder, however far the division is carried. In such cases one of three things may be done: The division may be carried as far as desired and the remainder dropped, the quotient may be obtained as far as desired and the remainder noted, or the remainder and divisor may make a fraction to be attached to the last quotient figure.

Divide 185.72 by 345, obtaining the quotient to three decimal places.

```
       .538
    -------
345)185.720
    172 5
    -----
     13 22
     10 35
     -----
      2 870
      2 760
      -----
        110
```

$^{110}/_{345} = ^{22}/_{69}$

1st Quotient .538 +.

2nd Quotient .538 Rem. .110.

3rd Quotient $.538^{22}/_{69}$.

Expressions like the third quotient are called complex decimals.

Divide carrying the quotient to two decimal places and express the result in three ways as shown above:

1. 45.7 by 21.
2. 7695.83 by 492.
3. 925 by 364.
4. 904.68 by 283.
5. 3.59 by 87.
6. 3658.74 by 364.

To Divide a Decimal by one with one or more Ciphers annexed.

One way of dividing a fraction by a whole number is to multiply the denominator by the whole number, leaving the numerator unchanged. This may be applied in dividing a decimal by one with ciphers annexed as follows:

RULE: To divide a decimal by one with one or more ciphers annexed move the decimal point as many places to the left as there are ciphers annexed to the one.

Divide 267.43 by 100. Answer 2.6743.

Divide:

1. 639.4 by 10.
2. 47.085 by 1000.
3. .0038 by 100.
4. 286. by 100.
5. 76.32 by 1000.
6. 4.694 by 1000000.

To Divide a Decimal by any number with ciphers annexed, cross off the ciphers at the right of the divisor and move the decimal point in the dividend as many places to the left, then divide by the remaining figures in the divisor as heretofore.

It is best not to move the decimal point, but to indicate by a mark the place to which it should be moved. Do not fail to place the mark before beginning the division and to place the decimal point in the quotient as soon as the mark in the dividend is reached.

Divide 2973.6 by 1800.

```
       1.65 2
      -------
18ØØ)29'73.6
     18
     --
     11 7
     10 8
     ----
        93
        90
        --
         3 6
         3 6
        ----
```

Divide:

1. 679.35 by 15000.
2. 7.84 by 39200.
3. .0018 by 4500.
4. 5.736 by 16000.
5. 57548.4 by 5460.
6. 72.9 by 270000.

To Reduce a Common Fraction to a Decimal.

A fraction is an indicated division in which the denominator is the divisor. Hence a fraction may be reduced to a decimal by annexing ciphers to the numerator and dividing by the denominator. A decimal point should be placed after the numerator before the ciphers are annexed.

Reduce $7/16$ to a decimal:

```
     .4375  Ans.
    ------
16)7.0000
   6 4
   ---
     60
     48
     --
     120
     112
     ---
       80
       80
       --
```

Reduce to decimals:

1.	$5/8$.	7.	$29/4$.
2.	$3/25$.	8.	$49/16$.
3.	$7/80$.	9.	$753/250$.
4.	$29/250$.	10.	$33/64$.
5.	$11/32$.	11.	$11/200$.
6.	$13/625$.	12.	$16/500$.

Most fractions will not make exact decimals. In reducing such fractions to decimals the result may be obtained to the desired number of decimal places and the remainder dropped or the result may be written as a complex decimal.

Reduce the following fractions to decimals of three decimal places and drop the remainder:

1.	$4/7$.	6.	$3/475$.
2.	$1/13$.	7.	$384/11$.
3.	$5/17$.	8.	$7/85$.
4.	$5/63$	9.	$27/65$
5.	$77/19$.	10.	$5/187$.

Reduce the following to complex decimals of four decimal places:

1.	$5/17$.	6.	$7/11000$.
2.	$39/19$.	7.	$700/23$.
3.	$81/41$.	8.	$3000/29$.
4.	$5/77$.	9.	$4/61$.
5.	$3/700$.	10.	$79/13$.

A complex decimal may be extended to any required number of decimal places by annexing ciphers to the numerator of the common fraction and dividing by its denominator and annexing the result to the original decimal part.

Reduce the $6.7\frac{5}{13}$ to a complex decimal of three decimal places.

Two additional places are required so two ciphers are annexed to the 5.

$$
\begin{array}{r}
.38\frac{6}{13} \\
\hline
13)5.00 \\
3\ 9 \\
\hline
1\ 10 \\
1\ 04 \\
\hline
6
\end{array}
$$

Hence $6.7\frac{5}{13} = 6.738\frac{6}{13}$.

Reduce the following to complex decimals of four decimal places:

1. $.63\frac{4}{11}$.
2. $7.8\frac{9}{77}$.
3. $67\frac{3}{17}$.
4. $.00\frac{4}{51}$.
5. $.760\frac{9}{34}$.
6. $.14\frac{3}{70}$.
7. $9.0\frac{20}{29}$.
8. $48\frac{7}{31}$.
9. $.0\frac{5}{300}$.
10. $60.27\frac{3}{47}$.

In complex decimals the common fraction belongs to the order to which it is annexed. Thus, $.26\frac{2}{3}$ means .26 and $\frac{2}{3}$ hundredths. $.0\frac{4}{7}$ is read $\frac{4}{7}$ tenths. A fraction does not represent an order and should never be immediately preceded by a decimal point.

Write $\frac{2}{3}$ hundredths; $4\frac{2}{7}$ thousandths; $\frac{5}{9}$ tenths.

A complex decimal may be reduced to a common fraction by reducing the numerator to an improper fraction and writing for the denominator the denominator of the common fraction followed by as many ciphers as there are decimal places.

Reduce $.47\frac{3}{7}$ to a common fraction.

$47\frac{3}{7} = \frac{332}{7}$, hence

$.47\frac{3}{7} = \frac{332}{700} = \frac{83}{175}$.

Reduce to common fractions:

1. $.63\frac{1}{3}$.
2. $.055\frac{5}{9}$.
3. $.63\frac{7}{11}$.
4. $8.07\frac{1}{9}$.
5. $.00\frac{4}{7}$
6. $3.0\frac{5}{11}$.
7. $.30\frac{4}{13}$.
8. $.6\frac{20}{21}$
9. $.4\frac{7}{30}$.
10. $.16\frac{3}{7}$.

The number of decimal places in a complex decimal may be reduced by treating the part which follows the desired stopping place as a complex decimal.

Reduce $.0286\frac{2}{3}$ to a complex decimal of two decimal places.

It is required to reduce the $.0086\frac{2}{3}$ to a fraction of hundredths.

$.86\frac{2}{3} = \frac{260}{300} = \frac{13}{15}$, hence

$.0286\frac{2}{3} = .02\frac{13}{15}$.

Note that .0086 is $.86\frac{2}{3}$ of a hundredth.

Reduce to complex decimals of one decimal place:

1. $6.73\frac{1}{3}$.
2. $.436\frac{3}{7}$.
3. $.07\frac{3}{11}$.
4. $.80\frac{5}{9}$.
5. $.340\frac{4}{7}$
6. $7.600\frac{25}{26}$.
7. $4.581\frac{9}{11}$.
8. $.000\frac{8}{13}$.
9. $.7930\frac{20}{27}$.
10. $8.47\frac{3}{11}$.

The student should be able to determine by inspection whether or not a fraction whose denominator is less than one hundred, or even one thousand will make an exact decimal.

A common fraction is reduced to a decimal by division. The numerator with ciphers annexed must exactly contain the denominator, or, what is the same thing, every factor of the denominator must cancel similar factors in the numerator or the quotient will be fractional. Annexing a cipher to the numerator introduces the prime factors 2 and 5 only, and additional ciphers merely increase the number of 2's and 5's.

Determine whether or not $\frac{9}{16}$, $\frac{9}{28}$, and $\frac{9}{125}$ will make exact decimals.

$$\frac{9}{16} = \frac{\overset{3\times3\times\cancel{2}\times5\times\cancel{2}\times5\times\cancel{2}\times5\times\cancel{2}\times5}{9.0000}}{\underset{\cancel{2}\times\cancel{2}\times\cancel{2}\times\cancel{2}}{16}} = .5625.$$

$$\frac{9}{28} = \frac{\overset{3\times3\times2\times5\times2\times5\times2\times5}{9.000}}{\underset{2\times2\times7}{28}}$$

The factor 7 cannot be introduced by annexing ciphers, hence $\frac{9}{28}$ will not make an exact pure decimal.

$$\frac{9}{125} = \frac{\overset{3\times3\times2\times5\times2\times5\times2\times5}{9.000}}{\underset{5\times5\times5}{125}} = .072$$

Determine by inspection which of the following fractions will make exact pure decimals, and test your conclusions:

1. $\frac{7}{18}$.	7. $\frac{27}{56}$.
2. $\frac{11}{24}$.	8. $\frac{33}{64}$.
3. $\frac{15}{32}$.	9. $\frac{17}{88}$.
4. $\frac{15}{25}$.	10. $\frac{23}{80}$.
5. $\frac{29}{45}$.	11. $\frac{26}{75}$.
6. $\frac{25}{48}$.	12. $\frac{6}{125}$.

If the fraction is in its lowest terms, the denominator must contain no prime factor except 2's or 5's, or 2's and 5's in order to reduce to a pure decimal. The number of decimal places is the same as the number of 2's or 5's in the denominator. The powers of 2 under 1000 are 2, 4, 8, 16, 32, 64, 128, 256, 512; those of 5 are 5, 25, 125, and 625. These numbers, or these with one or more ciphers annexed, are the only numbers under 1000 which being used as denominators of fractions will make pure decimals if the fractions are in their lowest terms.

Write all the denominators under 1000 of fractions that will make pure decimals and tell how many decimal places will be required for each. Test your answers.

When complex decimals are to be added or subtracted they must first be reduced to the same order.

Add $47\frac{5}{6}$ and $.27\frac{4}{7}$ and $.0954\frac{4}{21}$ and $8.643\frac{9}{14}$.

			21
47 $\frac{5}{6}$	=	$47.8333\frac{1}{3}$	7
.27 $\frac{4}{7}$	=	$.2757\frac{1}{7}$	3
$.0954\frac{4}{21}$	=	$.0954\frac{4}{21}$	4
8.643 $\frac{9}{14}$	=	$8.6436\frac{3}{7}$	9
		$56.8481\frac{2}{21}$	$\frac{23}{21}$

Add:

1. $.45\frac{7}{11} + 83\frac{5}{8} + .973\frac{1}{22} + 7.324$.

2. $\frac{47}{18} + .7\frac{4}{11} + .6281\frac{2}{3} + 6.03\frac{4}{15}$.

Add and subtract as the signs indicate:

1. $43.7\frac{3}{7} - .437\frac{3}{7}$.

2. $29.6\frac{5}{9} + .296\frac{5}{9} - 296\frac{5}{9}$.

3. $874\frac{2}{75} - 87.4\frac{2}{75} - .874\frac{2}{75} - 8.74\frac{2}{75}$.

4. $49.8\frac{3}{80} - .498\frac{3}{80} + 498\frac{3}{80}$

5. $23\frac{5}{9} + 2.35\frac{2}{3} + 23.7\frac{4}{9} - .2375\frac{5}{6}$.

To Multiply a Decimal by a Decimal.

In multiplying common fractions the numerators are multiplied together for the numerator and the denominators for the denominator o the product.

In a decimal the denominator is one with a cipher or ciphers annexed. To multiply two such numbers together we merely annex the ciphers of one number to the other. Thus $100\times1000=100000$. In multiplying two decimals together then the number of decimal places in the product equals the sum of the decimal places in the two factors.

To multiply a decimal by a decimal multiply as in whole numbers and point off as many decimals in the product as there are decimal places in both multiplicand and multiplier.

Multiply .4635 by .093.

$$\begin{array}{r} .4635 \\ .093 \\ \hline 13905 \\ 41715 \\ \hline .0431055 \end{array}$$

Multiply:

1. .784 by .932.
2. .2653 by .317.
3. .0695 by .01082.
4. 97.65 by .6438.
5. .03836 by .837.
6. .0907 by 470.

Complex decimals are multiplied in similar manner.

Multiply $67.954\frac{5}{7}$ by .09.

$$\begin{array}{r} 67.954\frac{5}{7} \\ .09 \\ \hline 6.11592\frac{3}{7} \end{array}$$

Multiply 7.65 by $.073\frac{5}{6}$

$$\begin{array}{r} 7.65 \\ .073\frac{5}{6} \\ \hline 6)3825 \\ \hline 637\frac{1}{2} \\ 2295 \\ 5355 \\ \hline .56482\frac{1}{2} \text{ or } .564825 \end{array}$$

Multiply $6.743\frac{2}{3}$ by $.28\frac{4}{7}$.

$$\begin{array}{rl} .6743\ \frac{2}{3} & \\ .28\ \frac{4}{7} & \\ \hline 7)\ \ 26974\ \frac{2}{3} & |\ 21 \\ \hline 3853\frac{11}{21} & |\ 11 \\ 18\ \frac{2}{3} & |\ 14 \\ 53944 & | \\ 13486 & | \\ \hline .192676\ \frac{4}{21} & |\frac{25}{21} \end{array}$$

Multiply:

1. $86.357\frac{4}{9}$ by .08.
2. $.0362\frac{7}{11}$ by 4.5.
3. 6.947 by $.004\frac{3}{7}$.
4. 463.85 by $18.5\frac{5}{11}$.
5. $1.058\frac{2}{3}$ by $.07\frac{4}{7}$.
6. $2.53\frac{5}{12}$ by $1.6\frac{5}{6}$.

To Divide a Decimal by a Decimal.

If the divisor and dividend have the same number of decimal places they will have a common denominator, and the quotient be a whole number the same as in division of common fractions.

Divide 4.688 by .293.

$$\begin{array}{r} 16 \\ \hline .293)4.688 \\ 2\ 93 \\ \hline 1\ 758 \\ 1\ 758 \\ \hline \end{array}$$

If the dividend has more decimal places than the divisor, a mark should be placed in the dividend cutting off as many decimal places as the divisor contains. This will determine he place where the intergral part of the quotient ends and the decimal part begins. The mark should be placed in the dividend before the division is begun.

Divide 4.7875 by .125.

$$\begin{array}{r} 38.3 \\ \hline .125)4.787`5 \\ 3\ 75\phantom{`5} \\ \hline 1\ 037 \\ \hline 375 \\ 375 \\ \hline \end{array}$$

Care should be taken that the quotient is placed in its proper position above the dividend and that the decimal point is placed in the quotient as soon as the mark in the dividend is reached. The decimal point in the quotient will come directly over the mark in the dividend if these precautions are observed. There should be a figure in the quotient over each figure used in the dividend to the right of the mark.

If the dividend contains fewer decimal places than the divisor has ciphers should be annexed to the dividend before beginning the division.

Divide 1137.6 by .237.

$$\begin{array}{r} 4\ 800 \\ \hline .237)1137.600` \\ 948\phantom{.600`} \\ \hline 189\ 6 \\ \hline \end{array}$$

These directions are summed up in the rule: **To divide by a decimal, Mark off as many decimal places in the dividend as there are decimal places in the divisor, beginning at the decimal point. Divide as in whole numbers, placing each figure of the quotient directly over the right hand figure of the dividend used in obtaining it. Place the decimal point in the quotient as soon as the mark in the dividend is reached.**

Divide:

1. 21.76 by .32.
2. 4.462 by .046.
3. .1005 by .067.
4. .4984 by 5.6.
5. .051 by .85.
6. 395.52 by .0309.
7. 3495.9 by .0215.
8. 9.7696 by .172.
9. .16854 by .00795.
10. 2280.96 by .0324.

A complex decimal is handled in division nearly the same as in mixed numbers.

Divide $4.678\frac{3}{7}$ by .08.

$$
\begin{array}{l}
\quad 58.4\frac{45}{56} \\
\hline
.08)4.67'8\ \frac{3}{7}
\end{array}
$$

(Side Work:)

$6\frac{3}{7} = \frac{45}{7}$.

$\frac{45}{7} \div 8 = \frac{45}{56}$.

Divide:

1. $4.8973\frac{5}{9}$ by .7.
2. $.3064\frac{7}{11}$ by .009.
3. $.00639\frac{5}{6}$ by 15.
4. $.00429\frac{2}{3}$ by 600.
5. $.3097\frac{3}{7}$ by 2.7.
6. $265.384\frac{4}{9}$ by 7000.

If the dividend of a complex decimal does not contain as many decimal places as the divisor has, carry the common fraction out as many additional decimal places as are desired.

Divide $4.83\frac{5}{12}$ by .0017.

$\frac{5}{12} = .41\frac{2}{3}$, hence

$4.83\frac{5}{12} = 4.8341\frac{2}{3}$.

$$
\begin{array}{l}
\qquad 2843\frac{32}{51} \\
\hline
.0017)4.8341'\ \frac{2}{3} \\
\qquad 3\ 4 \\
\hline
\qquad 1\ 43 \\
\qquad 1\ 36 \\
\hline
\qquad\quad 74 \\
\qquad\quad 68 \\
\hline
\qquad\quad 61 \\
\qquad\quad 51 \\
\hline
\qquad\quad 10\ \frac{2}{3}
\end{array}
$$

(Side Work)

$10\frac{2}{3} = \frac{32}{3}$

$\frac{32}{3} \div 17 = \frac{32}{51}$

Divide:

1. $76.5\frac{7}{11}$ by .023.
2. $.07\frac{5}{14}$ by .7365.
3. $83\frac{5}{6}$ by 6.295.
4. $24.0\frac{4}{9}$ by .0034.

If the divisor is a complex decimal reduce both dividend and divisor to fractions of the same denominator.

Divide .6795 by $2.43\frac{2}{9}$.

$$
\begin{array}{ll}
2.43\frac{2}{9} & .6795 \\
\times 9 & \times 9 \\
\hline
21.89 & 6.1155
\end{array}
$$

$$
\begin{array}{l}
\qquad .27\frac{2052}{2189} \\
\hline
21.89)6.11'55 \\
\qquad 4\ 378 \\
\hline
\qquad 1\ 7375 \\
\qquad 1\ 5323 \\
\hline
\qquad\quad 2052
\end{array}
$$

Divide $.0792\frac{5}{9}$ by $.174\frac{1}{3}$.

$$
\begin{array}{ll}
.174\frac{1}{3} & .0792\frac{5}{9} \\
\times 9 & \times 9 \\
\hline
1.569 & .7133
\end{array}
$$

$$
\begin{array}{l}
\qquad .45\frac{725}{1569} \\
\hline
1.569).713'30 \\
\qquad 627\ 6 \\
\hline
\qquad\ 85\ 70 \\
\qquad\ 78\ 45 \\
\hline
\qquad\quad 7\ 25
\end{array}
$$

Divide

1. 6.573 by .07$\frac{3}{7}$.
2. 84.3 by .29$\frac{2}{3}$.
3. .573 by 2.8$\frac{4}{11}$.
4. .643$\frac{3}{11}$ by 4$\frac{7}{11}$.
5. 1.2195$\frac{5}{7}$ by 2.3$\frac{2}{3}$.
6. 6.3$\frac{5}{12}$ by .361$\frac{1}{3}$.

If there is a remainder after the quotient has been carried the desired number of decimal places the result may be written as a complex decimal, the remainder may be dropped, or it may be noted and retained, care being taken to place the decimal point properly.

Divide 47.6 by 370, carrying the result to three decimal places and giving the remainder.

```
        .1 28
      -------
370̸)47.60
        37
      ----
        10 6
         7 4
        -----
         3 20
         2 96
        -----
          .24 Remainder
```

In the following examples carry the result to three decimal places and state what the remainder is.

Divide

1. .027 by 5.6.
2. 7.6 by .014.
3. .0068 by .235.
4. 900 by .0013.
5. .37 by 170.
6. 29.5 by .093.

Miscellaneous Exercises in Division.

Divide

1. 100 by .001.
2. .0003 by 3000.
3. 3240 by .027.
4. .00796 by 500.
5. .96064 by .32.
6. 425.92 by .605.
7. 3.4356 by 40.9.
8. 9101.57 by .0007.
9. 6660 by .074.
10. 6.1472 by 6.8.
11. 4.67$\frac{3}{14}$ by .33.
12. 2.78142 by 3.07.
13. .265 by 6.7$\frac{5}{9}$.
14. 2.322 by 86.
15. .0003 by 1.
16. .0022 by 200.

FRACTIONAL PARTS.

(1). To find $\frac{3}{4}$ of a number, divide it by 4 to get $\frac{1}{4}$ of it, and multiply the quotient by 3.

Thus to find $\frac{3}{4}$ of 24.

$\frac{1}{4}$ of 24=6.

Then $\frac{3}{4}$ of 24=18.

Other fractional parts of numbers may be found in a similar manner.

Find $\frac{5}{7}$ of 56.

$\frac{1}{7}$ of 56=8.

Then $\frac{5}{7}$ of 56=40.

Find:

1. $\frac{4}{9}$ of 63.
2. $\frac{7}{8}$ of 32.
3. $\frac{3}{7}$ of 84.
4. $\frac{5}{9}$ of 126.
5. $\frac{4}{3}$ of 36.
6. $\frac{8}{5}$ of 70.
7. $\frac{5}{6}$ of 50.
8. $\frac{7}{3}$ of 21.
9. $\frac{3}{8}$ of 42.

This is virtually multiplying the number by the fraction denoting the part to be found.

Find $\frac{6}{7}$ of 84.

$\frac{6}{7}$ of 84=72 or $\frac{6}{7}\times 84=72$.

Find by the last process:

1. $\frac{5}{9}$ of 54.
2. $\frac{7}{11}$ of 88.
3. $\frac{5}{12}$ of 36.
4. $\frac{2}{3}$ of 14.
5. $\frac{4}{7}$ of 30.
6. $\frac{4}{5}$ of 67.
7. $\frac{3}{5}$ of $\frac{4}{11}$.
8. $\frac{6}{7}$ of 5$\frac{1}{4}$.
9. $\frac{11}{8}$ of 7$\frac{2}{9}$.

(2). Find a number $\frac{2}{3}$ larger than 24. To do this find $\frac{2}{3}$ of the number (24 in this example) and add it to the number itself.

$\frac{2}{3}$ of 24=16.

24+16=40 Ans.

Find a number:

1. $\frac{1}{5}$ larger than 65.
2. $\frac{3}{7}$ larger than 42.
3. $\frac{8}{9}$ larger than 63.
4. $\frac{13}{8}$ larger than 56.
5. $\frac{5}{6}$ larger than 35.
6. $\frac{2}{7}$ larger than 245$\frac{5}{8}$.

This class of problems may also be solved as follows:

Find a number $\frac{4}{7}$ larger than 63.

A number is once itself. Once a number plus $\frac{4}{7}$ of it=$\frac{11}{7}$ of it. $\frac{11}{7}$ of 63=99.

Find a number:

1. $\frac{2}{9}$ larger than 72.
2. $\frac{5}{12}$ larger than 30.
3. $\frac{3}{5}$ larger than 27.
4. $\frac{9}{4}$ larger than 28.
5. $\frac{8}{3}$ larger than $\frac{12}{25}$.
6. $\frac{4}{7}$ larger than $3\frac{2}{3}$.

(3). Find a number $\frac{1}{5}$ smaller than 45.

$\frac{1}{5}$ of 45=9.

45—9=36.

Find a number:

1. $\frac{2}{7}$ smaller than 49.
2. $\frac{3}{11}$ smaller than 66.
3. $\frac{5}{9}$ smaller than 54.
4. $\frac{2}{5}$ smaller than 18.
5. $\frac{5}{8}$ smaller than $\frac{2}{3}$.
6. $\frac{1}{6}$ smaller than $46\frac{4}{5}$.

Problems like these may also be solved as follows:

Find a number $\frac{5}{12}$ less than 48 A number is once itself. Once a number minus $\frac{5}{12}$ of it leaves $\frac{7}{12}$ of it.

$\frac{7}{12}$ of 48 = 28.

Find a number

1. $\frac{4}{7}$ smaller than 77.
2. $\frac{7}{8}$ less than 40.
3. $\frac{3}{15}$ less than 120.
4. $\frac{2}{9}$ less than 60.
5. $\frac{2}{5}$ less than $\frac{7}{8}$.
6. $\frac{1}{3}$ less than $22\frac{5}{7}$.

PROBLEMS INVOLVING FRACTIONAL PARTS.

1. Henry has 56 marbles and James has $\frac{3}{4}$ as many. How many has James?

2. Mr. Mason had 42 tons of dried prunes in 1912. In 1913 his crop was $\frac{5}{6}$ as large. What was his crop in 1913?

3. He sold his prunes at \$70 a ton in 1912, and for $\frac{1}{5}$ more per ton in 1913. How much did he receive for each crop?

4. Raymond is 5 feet tall, Sherman is $\frac{1}{12}$ taller, and Homer is $\frac{4}{5}$ as tall as Sherman. How tall are Sherman and Homer, respectively? (Answer in feet and inches).

5. An apple tree bore 720 pounds of fruit and the crop on a peach tree was $\frac{1}{3}$ lighter. A cherry tree had a crop $\frac{1}{3}$ heavier than that on the peach tree. How many pounds did the cherry tree bear?

6. A hog is worth \$20. If a sheep is worth $\frac{3}{10}$ as much as a hog, and a goat is worth $\frac{3}{4}$ as much as a sheep, what is the value of 75 hogs, 350 sheep and 12 goats?

7. A boat can run 24 miles an hour, a passenger train can run $\frac{3}{4}$ faster, and an aeroplane can travel $\frac{5}{7}$ faster than the passenger train. How long will it take each to travel 1200 miles?

8. A cubic foot of fresh water weighs 1000 ounces and sea water is $\frac{1}{40}$ heavier. Find the weight in pounds of a cubic yard of sea water.

9. Cork is $\frac{3}{4}$ lighter than fresh water. Find the weight in pounds of 50 cubic feet of cork.

10. A tree 9 feet tall increased its height $\frac{1}{3}$ each year for four years. What was its height at the end of the fourth year?

(4). 35 is $\frac{5}{7}$ of what number?

If $\frac{5}{7}$ of the number=35

$\frac{1}{7}$ of the number= 7

$\frac{7}{7}$ of the number=49

1. 48 is $\frac{3}{8}$ of what number?
2. 36 is $\frac{4}{9}$ of what number?
3. 24 is $\frac{8}{3}$ of what number?
4. 54 is $\frac{3}{2}$ of that number?
5. 231 is $\frac{7}{3}$ of what number?
6. 15 is $\frac{4}{7}$ of what number?
7. 25 is $\frac{3}{5}$ of what number?
8. $\frac{12}{13}$ is $\frac{4}{11}$ of what number?
9. $8\frac{3}{4}$ is $\frac{5}{6}$ of what number?
10. $\frac{5}{8}$ is $\frac{3}{8}$ of what number?

The same result will be obtained in examples like those just given by dividing the number by the fraction representing the part.

105 is $\frac{7}{5}$ of what number?

$105 \div \frac{7}{5} = 105 \times \frac{5}{7} = 75$.

1. 30 is $\frac{5}{6}$ of what number?
2. 75 is $\frac{3}{8}$ of what number?
3. 144 is $\frac{16}{7}$ of what number?
4. 175 is $\frac{7}{5}$ of what number?
5. 27 is $\frac{4}{9}$ of what number?
6. 65 is $\frac{7}{13}$ of what number?
7. $8\frac{2}{5}$ is $\frac{7}{9}$ of what number?
8. $\frac{13}{14}$ is $\frac{3}{8}$ of what number?

(5). 30 is $\frac{1}{5}$ greater than what number? A number is $\frac{5}{5}$ of itself. Once a number plus $\frac{1}{5}$ of it equals $\frac{6}{5}$ of it.

$\frac{5}{5}+\frac{1}{5}=\frac{6}{5}$

$\frac{6}{5}$ of the number=30

$\frac{1}{5}$ of the number=5

$\frac{5}{5}$ of the number=25 Ans.

1. 84 is $\frac{1}{6}$ greater than what no.?
2. 45 is $\frac{2}{3}$ greater than what no.?
3. 56 is $\frac{3}{4}$ greater than what no.?
4. 120 is $\frac{3}{5}$ greater than what no.?
5. 18 is $\frac{1}{6}$ greater than what no.?
6. $8\frac{4}{7}$ is $\frac{1}{4}$ greater than what no.?
7. $5\frac{1}{2}$ is $\frac{2}{7}$ greater than what no.?
8. $\frac{18}{25}$ is $\frac{1}{5}$ greater than what no.?

The work may be shortened by division. 120 is $\frac{3}{5}$ greater than what number?

$\frac{5}{5}+\frac{3}{5}=\frac{8}{5}$

$\frac{8}{5}$ of the number = 120.

$120 \div \frac{8}{5} =$ 75 Ans.

1. 126 is $\frac{2}{7}$ greater than what no.?
2. 264 is $\frac{3}{8}$ greater than what no.?
3. 560 is $\frac{3}{7}$ greater than what no.?
4. 585 is $\frac{4}{9}$ greater than what no.?
5. 180 is $\frac{3}{5}$ greater than what no.?
6. $26\frac{2}{3}$ is $\frac{1}{3}$ greater than what no.?
7. $10\frac{1}{2}$ is $\frac{2}{9}$ greater than what no.?
8. $\frac{24}{25}$ is $\frac{5}{7}$ greater than what no.?

(6). 56 is $\frac{3}{7}$ less than what number? A number is $\frac{7}{7}$ of itself, and if $\frac{3}{7}$ of it is taken away, $\frac{4}{7}$ will be left.

$\frac{7}{7}-\frac{3}{7}=\frac{4}{7}$.

$\frac{4}{7}$ of the number=56.

$\frac{1}{7}$ of the number=14.

$\frac{7}{7}$ of the number=98 Ans.

1. 18 is $\frac{1}{3}$ less than what no.?
2. 120 is $\frac{3}{8}$ less than what no.?
3. 252 is $\frac{2}{9}$ less than what no?
4. 84 is $\frac{1}{7}$ less than what no.?
5. $7\frac{5}{7}$ is $\frac{2}{3}$ less than what no.?
6. $\frac{42}{65}$ is $\frac{4}{9}$ less than what no.?

The process may be shortened by division. 75 is $\frac{2}{5}$ less than what number?

$\frac{5}{5}-\frac{2}{5}=\frac{3}{5}$.

$\frac{3}{5}$ of the number= 75.

$75 \div \frac{3}{5}=125$ Ans.

1. 126 is $\frac{2}{9}$ less than what no.?
2. 48 is $\frac{4}{7}$ less than what no.?
3. 36 is $\frac{3}{4}$ less than what no.?
4. 25 is $\frac{2}{5}$ less than what no.?
5. $5\frac{1}{7}$ is $\frac{5}{8}$ less than what no.?
6. $\frac{4}{5}$ is $\frac{2}{9}$ less than what no.?

PROBLEMS.

1. Mary has 45 cents, which is $\frac{3}{4}$ of what Helen has. How much has Helen?

2. John has $\frac{3}{4}$ as many marbles as Herbert and they have together 56 marbles. How many has each?

3. Mr. Morgan raised 240 tons of prunes and his crop was $\frac{2}{3}$ larger than Mr. Payne's. What was Mr. Payne's crop?

4. The distance from Palo Alto to San Francisco is $\frac{5}{6}$ greater than the distance from San Jose to Palo Alto and the distance from San Jose to San Francisco is 51 miles. What is the distance from San Jose to Palo Alto?

5. An orchard consisting of peach trees and apricot trees contains 800 trees. There are $\frac{2}{3}$ more peach trees than apricot trees. How many of each kind?

6. Mary is 42 inches tall and she is $\frac{3}{11}$ shorter than Hannah. How tall is Hannah?

7. An oak was 75 feet tall. It was $\frac{2}{3}$ taller than a walnut and the walnut was $\frac{2}{3}$ shorter than an eucalyptus. How tall was the eucalyptus?

(7). 18 is what part of 24?

1 is $\frac{1}{24}$ of 24.

18 is $\frac{18}{24}$ of 24.

$\frac{18}{24}=\frac{3}{4}$. Hence 18 is $\frac{3}{4}$ of 24.

As a rule the result is obtained directly by writing the result in fractional form at once and reducing to its lowest terms.

$\frac{18}{24}=\frac{3}{4}$. Hence 18 is $\frac{3}{4}$ of 24.

28 is what part of 35?

$\frac{28}{35}=\frac{4}{5}$. Hence 28 is $\frac{4}{5}$ of 35.

42 is what part of 32?

$\frac{42}{32}=\frac{21}{16}$. Hence 42 is $\frac{21}{16}$ of 32.

What part of 18 is 27?

$\frac{27}{18}=\frac{3}{2}$. Hence 27 is $\frac{3}{2}$ of 18.

Notice that the part, which becomes the numerator of the fraction, is the subject of the sentence, and that the whole, which becomes the denominator of the fraction, is the object of the preposition *of*.

1, 28 is what part of 42?
2. 36 is what part of 63?
3. 64 is what part of 48?
4. What part of 16 is 26.
5. 8 inches is what part of 3 feet?
6. 125 rd. is what part of a mile?

Fractions are handled in a similar manner. $2\frac{3}{4}$ is what part of 22?

$\frac{2\frac{3}{4}}{22}=\frac{11}{88}=\frac{1}{8}$.

Hence $2\frac{3}{4}$ is $\frac{1}{8}$ of 22.

$\frac{6}{7}$ is what part of $5\frac{1}{3}$?

$\frac{\frac{6}{7}}{5\frac{1}{3}}=\frac{6}{7}\times\frac{3}{16}=\frac{9}{56}$ Ans.

1. $4\frac{2}{3}$ is what part of $6\frac{2}{7}$?
2. $\frac{5}{9}$ is what part of $1\frac{7}{8}$?
3. $\frac{2}{3}$ ft. is what part of $\frac{2}{7}$ yd?
4. What part of $3\frac{2}{3}$ is $1\frac{3}{4}$?
5. What part of $1\frac{5}{7}$ is $4\frac{2}{7}$?
6. What part of $\frac{7}{9}$ yd. is $\frac{3}{4}$ ft.?

(8). 63 is what part greater than 36?

It is necessary to find how much it is greater.

$63-36=27$, $\frac{27}{36}=\frac{3}{4}$. Hence 63 is $\frac{3}{4}$ greater than 36.

54 is what part greater than 42?

$54-42=12$, $\frac{12}{42}=\frac{2}{7}$, hence 54 is $\frac{2}{7}$ greater than 42.

Notice that the difference between the two numbers is the part and becomes the numerator of the fraction, and that the whole, which becomes the denominator of the fraction, follows *than*.

75 is what part greater than 45?

$75-45=30$, $\frac{30}{45}=\frac{2}{3}$. Hence 75 is $\frac{2}{3}$ greater than 45.

1. 60 is what part greater than 42?
2. 75 is what part greater than 54?
3. 180 is what part greater than 80?
4. $\frac{15}{16}$ is what part greater than $\frac{1}{2}$?
5. $9\frac{3}{4}$ is what part greater than $5\frac{1}{2}$?
6. $\frac{6}{7}$ is what part greater than $\frac{4}{11}$?

(9). 32 is what part less than 44?

It is necessary to find how much 32 is less than 44 first. $44-32=12$; $\frac{12}{44}=\frac{3}{11}$. Hence 32 is $\frac{3}{11}$ less than 44.

36 is what part less than 56?

$56-36=20$, $\frac{20}{56}=\frac{5}{14}$. Hence the answer is $\frac{5}{14}$.

Notice that the difference between the numbers becomes the numerator of the fraction, and that the whole, which becomes the denominator, follows the word *than*.

64 is what part less than 92?

$92-64=28$, $\frac{28}{92}=\frac{7}{23}$, Ans.

1. 56 is what part less than 77?
2. 45 is what part less than 75?
3. 32 is what part less than 84?
4. 23 is what part less than 31?
5. $\frac{5}{9}$ is what part less than $\frac{2}{3}$?
6. $3\frac{4}{7}$ is what part less than $5\frac{5}{7}$?

MISCELLANEOUS EXERCISES IN FRACTIONAL PARTS.

1. Find (1) $\frac{3}{4}$ of 84; (2) a number $\frac{3}{4}$ greater than 84; (3) a number $\frac{3}{4}$ less than 84.

2. (1) 84 is $\frac{3}{4}$ of what number? (2) $\frac{3}{4}$ greater than what number? (3) $\frac{3}{4}$ less than what number?

3. (1) 56 is what part of 84? (2) what part less than 84? (3) 84 is what part greater than 56?

4. $\frac{3}{4}$ of 84 is $\frac{3}{4}$ greater than what number?

5. $\frac{3}{4}$ of 84 is $\frac{3}{4}$ less than what number?

6. Find a number $\frac{11}{3}$ larger than 126.

7. 126 is $\frac{11}{3}$ larger than what number?

8. 75 is $\frac{2}{5}$ smaller than what number?

9. Find a number $\frac{2}{5}$ smaller than 75.

10. 100 is what part larger than 75?

PROBLEMS IN FRACTIONAL PARTS.

1. Mr. Stone bought a horse for \$120, and sold it for $\frac{4}{3}$ of its cost. How much did he gain?

2. Mr. Andrews sold a house for \$4800, which was $\frac{4}{3}$ of its cost. What was his gain?

3. Mr. Brown sold a bicycle for \$24, which was $\frac{1}{3}$ less than its cost. What was his loss?

4. A pole 28 feet long was broken so that the length of one part was $\frac{3}{4}$ of the length of the other. Find length of each part.

Let $\frac{4}{4}$ = the length of the longer part.
Then $\frac{3}{4}$ = the length of the shorter part.
And $\frac{7}{4}$ = the length of the whole.
$\frac{7}{4}$ of longer part=28 ft.
$\frac{1}{4}$ of longer part= 4 ft.
$\frac{4}{4}$ of longer part=16 ft. longer part.
$\frac{3}{4}$ of longer part=12 ft. shorter part.

5. The sum of two numbers is 240 and the smaller is $\frac{3}{5}$ of the larger. Find each number.

6. The difference between two numbers is 84, and the smaller is $\frac{4}{7}$ of the larger. Find each number.

7. The sum of three numbers is 132. The first is $\frac{3}{4}$ of the second, and the third is $\frac{3}{5}$ of the first. Find each number.

8. $\frac{2}{5}$ of Mary's age equals $\frac{2}{3}$ of Horace's, and the sum of their ages is 24 years. How old is each?

Since $\frac{2}{5}$ of Mary's age=$\frac{2}{3}$ of Horace's.
$\frac{1}{5}$ of Mary's age=$\frac{1}{3}$ of Horace's.
And $\frac{5}{5}$ of Mary's age=$\frac{5}{3}$ of Horace's.
Let $\frac{3}{3}$=Horace's age.
Then $\frac{5}{3}$=Mary's age.
And $\frac{8}{3}$=the sum of their ages.
Then $\frac{8}{3}$ of Horace's age=24 years
$\frac{1}{3}$ of Horace's age= 3 years
$\frac{3}{3}$ of Horace's age= 9 years Horace's age.
$\frac{5}{3}$ of Horace's age=15 years Mary's age.

9. An estate of \$27940 is to be shared by a brother and sister so that $\frac{2}{3}$ of the brother's share is equal to $\frac{4}{5}$ of the sister's. Find the share of each.

10. Mary has 48 inches of ribbon. She has $\frac{1}{3}$ more than Susan, and Susan has $\frac{2}{7}$ more than Jane. How many inches have Susan and Jane respectively?

Ans., S. 36 in.; J. 28 in.

11. There are 124 more boys than girls at a certain school, and $\frac{2}{3}$ of the boys equals $\frac{6}{7}$ of the girls. How many students are there?

12. A certain orchard consisting of apples, peaches and pear trees contains 3600 trees; $\frac{1}{3}$ of the apple trees equals $\frac{1}{4}$ of the peach trees, and $\frac{1}{4}$ of the peach trees equals $\frac{1}{5}$ of the pear trees. How many trees of each kind does the orchard contain?

13. Hannah is 60 inches tall. She is $\frac{1}{6}$ shorter than John, and John is $\frac{1}{3}$ taller than Emily. How tall is Emily? Ans., 54 in.

PERCENTAGE.

(1) To Express Per Cent as a Decimal.

Per cent means hundredths. If the per cent contains no decimal or fractional part, its is only necessary to point off two decimal places and omit the character (%), or word *per cent.*

27% = .27; 3 per cent = .03;
285% = 2.85

Express the following decimally:

1. 7%.
2. 65%.
3. 495%.
4. 2 per cent.
5. 4000 per cent.
6. 967 per cent.
7. 800%.
8. 40%.
9. 100%.
10. 5384%.
11. 478%.
12. 198%.

If the per cent contains a decimal part move the decimal point two places to the left and omit the character.

Express decimally:

1. .6%.
2. 4.7%.
3. .25%.
4. 46.5%.
5. 895.37%.
6. .003%.
7. .09 per cent.
8. 4.9 per cent.
9. .0079 per cent.

If the per cent contains a common fraction express it as a complex decimal and reduce the resulting expression to a pure or mixed decimal, if possible.

$5\frac{1}{8}\% = .05\frac{1}{8} = .05125.$
$\frac{3}{7}\% = .00\frac{3}{7}.$

Express decimally:

1.	$\frac{3}{5}\%$.	7.	$.7\frac{3}{40}\%$.
2.	$\frac{1}{25}\%$.	8.	$.0\frac{3}{125}\%$.
3.	$43\frac{5}{8}\%$.	9.	$200\frac{3}{5}\%$.
4.	$\frac{1}{4}\%$.	10.	$.6\frac{4}{11}\%$.
5.	$6\frac{1}{16}\%$.	11.	$268\frac{2}{3}\%$.
6.	$247\frac{1}{50}\%$.	12.	$.18\frac{5}{9}\%$.

(2) To Express Per Cent as a Common Fraction.

In general it is best to change the per cent to a decimal and then change that result to a common fraction and reduce to its lowest terms.

$15\% = .15 = \frac{15}{100} = \frac{3}{20};$
$3\frac{1}{3}\% = .03\frac{1}{3} = \frac{10}{300} = \frac{1}{30}.$

Express as a common fraction in its lowest terms:

1.	48%.	7.	260%.
2.	225%.	8.	$7\frac{1}{7}\%$.
3.	4.5%.	9.	$.0\frac{5}{9}\%$.
4.	.025%.	10.	$33\frac{1}{3}\%$.
5.	$.6\frac{1}{4}\%$.	11.	$.33\frac{1}{3}\%$.
6.	$.05\frac{5}{7}\%$.	12.	$3.3\frac{1}{3}\%$.

The work may be shortened by omitting the decimal form, being careful to annex the two ciphers to the denominator in place of the character, %.

$.24\% = \frac{24}{10000} = \frac{3}{1250};$
$9\frac{3}{7}\% = \frac{66}{700} = \frac{33}{350};$
$1.4\frac{1}{6}\% = \frac{85}{6000} = \frac{17}{1200}.$

Express as a common fraction in its lowest terms:

1.	$13\frac{1}{3}\%$.	6.	$4.5\frac{5}{11}\%$.
2.	$.6\frac{2}{3}\%$.	7.	$.57\frac{1}{7}\%$.
3.	$.0\frac{5}{7}\%$.	8.	$76\frac{2}{3}\%$.
4.	$7\frac{1}{7}\%$.	9.	$400\frac{5}{8}\%$.
5.	$22\frac{1}{9}\%$.	10.	$\frac{5}{7}\%$.

Some per cents freuqently used reduce to small fractions. Pupils should become so familiar with these that they will readily recognize them. For example, $25\% = \frac{1}{4}$; $50\% = \frac{1}{2}$; $75\% = \frac{3}{4}$.

Reduce to common fractions:

1. 10%, 20%, 30%, 40%, 50%, 60%, 70%, 80%, 90%.

2. $8\frac{1}{3}\%$, $16\frac{2}{3}\%$, $33\frac{1}{3}\%$, $66\frac{2}{3}\%$.

3. Count by $12\frac{1}{2}$ to 100, beginning with $12\frac{1}{2}$. Write the character % after each result and reduce to a common fraction in its lowest terms.

4. Count by $8\frac{1}{3}$ from $8\frac{1}{3}$ to 100, annexing the character % after each result and reduce as in exercise 3.

5. Count by $11\frac{1}{9}$ to 100 and treat the results as in exercise 3.

6. Count by $14\frac{2}{7}$ to 100 and treat the results as in exercise 3.

(3) To Find a Given Per Cent of a Number.

As a rule it is best to express the per cent as a decimal and multiply the given number by it.

Find 17% of 7658.

```
  7658
   .17
 -----
 53606
 7658
 -----
1301.86 Ans.
```

Find $7\frac{3}{5}$% of 47.85.

```
 47.85
  .076
 -----
 28710
3 3495
------
3.63660
```

Find $6\frac{4}{7}$% of 375.

```
   375
  .06 4/7
 -------
7)1500
 -------
  214 2/7
 2250
 -------
 24.64 2/7
```

Find by the method here used:

1. $\frac{2}{3}$% of 867.
2. 128% of 42.5.
3. 800% of 375.
4. 25% of 6.96.
5. $\frac{1}{4}$% of 7651.
6. $.8\frac{2}{9}$% of 7965.

If the per cent reduces to a small fraction it is usually best to use the fraction.

Find $16\frac{2}{3}$% of 894. $16\frac{2}{3}\%=\frac{1}{6}$; $\frac{1}{6}$ of 894=149 Ans.

Find $62\frac{1}{2}$% of 6.256. $62\frac{1}{2}=\frac{5}{8}$; $\frac{5}{8}$ of 6.256=3.910 Ans.

Find by using fractions:

1. 75% of 788.
2. $83\frac{1}{3}$% of 35.4.
3. $36\frac{4}{11}$% of 4.632.
4. $20\frac{1}{7}$% of 238.
5. $87\frac{1}{2}$% of 38.
6. $77\frac{7}{9}$% of $127\frac{2}{7}$.

Find by the easiest method:

1. $\frac{2}{7}$% of 6545.
2. $\frac{3}{8}$% of 6752.
3. $12\frac{1}{2}$% of 47.31.
4. 150% of $\frac{5}{8}$.
5. $\frac{3}{5}$% of $4\frac{2}{7}$.
6. $4\frac{3}{7}$% of $1.65\frac{5}{9}$.

(4). To find a number a given per cent larger than a given number.

To do this it is only necessary to find the per cent of the number and add this to the number.

Find a number 15% greater than 486.

15%=.15.

```
 486           486
 .15          72.90
 ----         ------
2430          558.90 Ans
486
-----
72.90
```

or since a number is 100% of itself, 100%+15%=115%.

```
   4 86
   1.15
  -----
  24 30
  48 6
 486
 ------
 558.90 Ans.
```

Find a number $16\frac{2}{3}$% greater than 593.

$16\frac{2}{3}\%=\frac{1}{6}$; $\frac{1}{6}$ of 593=$98\frac{5}{6}$.

593 plus $98\frac{5}{6}$=$691\frac{5}{6}$ Ans.

In what follows use whatever plan seems shortest.

Find a number:

1. 17% larger than 27.
2. 275% larger than 654.
3. $2\frac{3}{4}$% larger than 1440.
4. $66\frac{2}{3}$% larger than 216.
5. $\frac{4}{7}$% larger than 945.
6. $\frac{1}{8}$% larger than 674.

(5). **To find a number a given per cent less than a given number.**

The process is identical with that of (4) except that after the required per cent is found, decimally or otherwise, it is subtracted from the original number.

Find a number 24% less than 575.

24%=.24.

$$\begin{array}{r} 575 \\ .24 \\ \hline 2300 \\ 1150 \\ \hline 138.000 \end{array} \qquad \begin{array}{r} 575 \\ 138 \\ \hline 437 \text{ Ans.} \end{array}$$

or 100% — 24% = 76% = .76.

$$\begin{array}{r} 575 \\ .76 \\ \hline 3450 \\ 4025 \\ \hline 437.00 \text{ Ans.} \end{array}$$

Find a number $37\frac{1}{2}$% less than 645.

$37\frac{1}{2}$%=$\frac{3}{8}$; $\frac{3}{8}$ of 645=$241\frac{7}{8}$.

645—$241\frac{7}{8}$=$403\frac{1}{8}$ Ans.

100%—$37\frac{1}{2}$%=$62\frac{1}{2}$%=$\frac{5}{8}$;

$\frac{5}{8}$ of 645=$403\frac{1}{8}$ Ans.

In what follows, use the method which seems shortest.

Find a number:

1. 26% less than 182.
2. 3.7% less than 579.
3. $\frac{4}{9}$% less than 837.
4. $33\frac{1}{3}$% less than 548.
5. $6\frac{2}{3}$% less than 759.
6. $.0\frac{3}{7}$% less than 2681.

Problems.

1. Sea water is 2.7% heavier than fresh water, and a cubic foot of fresh water weighs $62\frac{1}{2}$ pounds. Find the weight of a cubic yard of sea water.

2. Ice is 7% lighter than fresh water. Find the weight of 100 cu-ft. of ice.

3. An orchard contains 600 peach trees. There are 60% more prune trees than peach trees; 60% fewer apple trees than peach trees, and the number of pear trees is 60% of the number of peach trees. How many trees does the orchard contain?

4. Mrs. Kammerer put 1200 eggs in an incubator. 95% of the eggs hatched and 60% of the chicks were pullets. She sold 75% of the roosters. How many chickens remain?

5. A tree 8 feet tall increased its height 25% each year for 4 years. What was its height at the end of the fourth year?

6. Mr. Burton bought a house for $4500 and afterward sold it at an advance of 10% on its cost. The purchaser sold it at an advance of 10% and the last purchaser sold at a reduction of 20%. What was the last selling price?

7. A city whose population was 1200 in 1900 increased in population 275% in ten years. At this rate of increase. what will its population be in 1920?

8. 2.55% of the weight of sea water is salt. How much salt can be obtained from 16000 cubic yards of sea water?

9. The waters of Salt Lake are 20% heavier than fresh water and 13% of this weight is common salt. The area of the lake is 2500 sq. miles, and its average depth is 20 ft. How many tons of salt does the lake contain?

(6) **To find a number when a certain per cent of it is given.**

In finding a given per cent of a number the number is multiplied by

the per cent expressed decimally or as a common fraction.

The per cent of a number, then, is the product and the number and the per cent are factors. Hence to find a number when a certain per cent of it is given, it is only necessary to divide the given per cent of the number by the per cent expressed either decimally or as a fraction.

834 is 15% of what number?

15%=.15.

```
     55 60
.15)834.00
```

826 is 87½% of what number?

87½% = ⅞.

⅞ of the number=826.

⅛ of the number=119.

8/8 of the number=952.

(or 826÷⅞=952.)

In the following exercises express the per cent decimally:

1. 731 is 17% of what number?
2. 378 is 135% of what number?
3. 1045 is 8½% of what number?
4. 684 is 4 2/7% of what number?
5. 12.25 is .3 1/9% of what number?
6. 8715 is 275% of what number?

In the following problems express the per cent as a fraction:

1. 483 is 87½% of what number?
2. 895 is 16⅔% of what number?
3. 764 is 44 4/9% of what number?
4. 287 is 37½% of what number?
5. 5.754 is 162½% of what number?
6. 687 is ⅔% of what number?

(7). **To find a number when a certain per cent more than it is given.**

Since any number is once itself, or 100% of itself, it is only necessary to add the given per cent, expressed either decimally or as a fraction to 1 similarly expressed, and divide the given number by this sum.

729 is 35% greater than what number?

100%+35%=135%;
135%=1.35.

```
        5 40  Ans.
     --------
1.35)729.00
     675
     ----
      54 0
      54 0
      ----
```

594 is 37½% greater than what number?

37½%=⅜; 1+⅜=11/8. 11/8 of number=594. 594÷11/8=432 Ans.

In the following use whichever method seems shortest:

1. 996 is 33⅓% more than what number?
2. 1764 is 22½% more than what number?
3. 29.07 is 112½% more than what number?
4. 1.397 is 46⅔% more than what number?
5. 601.5 is ¼% more than what number?
6. 28105 is ⅛% more than what number?

(8). **To find a number when a certain per cent less than it is given.**

Subtract given per cent from 100% and expressing the given result decimally, divide the given number by it.

396 is 45% less than what number?

100%—45%=55%; 55%=.55.

```
       7 20 Ans
    --------
.55)396.00
    385
    ----
     11 0
     11 0
     ----
```

56.73 is 57 1/7% less than what number?

100%—57 1/7%=42 6/7%;
42 6/7%=3/7.
3/7 of number=56.73.
56.73÷3/7=132.37.

Use the method that seems shortest in the following examples.

1. 103 is 15% less than what number?

2. 894 is $33\frac{1}{3}$% less than what number?

3. 4695 is $37\frac{1}{2}$% less than what number?

4. $\frac{6}{19}$ is $62\frac{1}{2}$% less than what number?

5. 173.92 is $2\frac{3}{7}$% less than what number?

6. 156.6 is $3\frac{1}{3}$% less than what number?

Problems.

1. 1173 is 15% of what number? 15% greater than what number? 15% less than what number?

2. Pine wood is 55% lighter than water. How many cubic feet in a ton of pine wood?

3. What is the weight of 1000 feet of pine lumber?

4. Mr. Reynolds had 25% more prunes than Mr. Leonard, and Mr. Woodham had 10% more than Mr. Reynolds. They had together 3996 tons. How much had each?

5. Mr. Munroe left $15290 to his three children, Henry, Mary and Susan. He gave Mary 30% more than Henry and Susan $12\frac{1}{2}$% less than Mary. How much did he give each?

6. Milk is 3.2% heavier than water. How much should a gallon (231 cu. in.) of milk weigh?

7. A tree increased its height 20% each year for four years and was then 54 feet tall. How tall was it at first?

8. Mrs. Stewart is 12% shorter than her husband and the sum of their height is $117\frac{1}{2}$ inches. Find the height of each.

9. Mr. Simpson sold a horse losing 10% of its cost. With the money he bought another which he sold at a loss of 10%. His total loss was $23.75. What did the first horse cost?

10. Mr. Johnson sold a house at a loss of 30% on the cost; with the money he bought another which he sold at a gain of 30% on its cost. His net loss was $432. Find the cost of each house.

(9). **To express a decimal as per cent.**

Since per cent means hundredths it is only necessary, if the number consists of two decimal places, to omit the decimal point and use the character %.

Thus, $.17=17\%$; $.03\frac{1}{7}=3\frac{1}{7}\%$; $3.67=367\%$.

If the expression contains less than two decimal places, extend it to two places, omit the point, and use the character as before.

Thus, $3.7=3.70=370\%$.

$.6\frac{3}{11}=.62\frac{8}{11}=62\frac{8}{11}\%$.

$4\frac{3}{5}=4.60=460\%$.

$.7\frac{3}{85}=.70\frac{6}{17}=70\frac{6}{17}\%$.

If there are more than two decimal places in the expression move the decimal point two places to the right and introduce the character.

Thus, $.364=36.4\%$ or $36\frac{2}{5}\%$;

$.0475=4.75\%$ or $4\frac{3}{4}\%$;

$.007\frac{2}{9}=.7\frac{2}{9}$ or $\frac{13}{18}\%$.

Express the following as per cent:

1.	$2.04\frac{2}{7}$.	9.	$\frac{7}{250}$.
2.	$20.4\frac{2}{7}$.	10.	$.000\frac{7}{250}$.
3.	$204\frac{2}{7}$.	11.	$.0\frac{7}{250}$.
4.	$.204\frac{2}{7}$.	12.	$.00\frac{7}{250}$.
5.	$.8\frac{3}{125}$.	13.	.0025.
6.	$8\frac{3}{125}$.	14.	225.
7.	$.008\frac{3}{125}$.	15.	.1225.
8.	$.08\frac{3}{125}$.	16.	12.25.

(10). **To express a common fraction as per cent.**

It is only necessary to reduce the fraction to a decimal of at least two decimal places and then use the character % instead of two decimal places.

Thus, $\frac{5}{18}=.27\frac{7}{9}=27\frac{7}{9}\%$.

$\frac{4}{625}=.0064=.64\%$.

$\frac{81}{40}=2.025=202.5\%$.

Express the following fractions as per cent:

1. $\frac{11}{25}$.
2. $\frac{163}{80}$.
3. $4\frac{1}{16}$.
4. $.9\frac{1}{4}$.
5. $3.7\frac{1}{2}$.
6. $\frac{3}{77}$.
7. $49\frac{2}{85}$.
8. $7.0\frac{5}{44}$.
9. $.0\frac{1}{7}$.
10. $.00\frac{1}{7}$.
11. $\frac{1}{7}$.
12. $.000\frac{1}{7}$.
13. $50\frac{3}{49}$.
14. $5.0\frac{3}{49}$.
15. $.50\frac{3}{49}$.
16. $.0\frac{1}{4}$.

(11). **To find what per cent one number is of another.**

It will be seen that this is similar to (7) of fractional parts. First express the relation in the form of a fraction, and then change the fraction to per cent.

15 is what per cent of 48?

$\frac{15}{48}=.3125=31.25\%$ or $31\frac{1}{4}\%$.

27 is what per cent of 67?

$\frac{27}{67}=.40\frac{20}{67}=40\frac{20}{67}\%$ Ans.

```
      .40 20/67
    ---------
 67)27.00
    26 8
    ----
      20
```

Find what per cent:

1. 65 is of 125.
2. 88 is of 160.
3. 629 is of 125.
4. $43\frac{4}{7}$ is of 19.
5. 84 is of $217\frac{3}{11}$.
6. 7 is of 1250.
7. $4\frac{1}{12}$ is of $16\frac{1}{3}$.
8. $59\frac{3}{7}$ is of $84\frac{5}{9}$.
9. 6.75 is of 27.5.
10. 7.4 is of 2.75.
11. .674 is of 25.
12. 56.4 is of 25.

(12). **To find what per cent one number is greater than another.**

This is similar to (8) in fractional parts. First find how much one number is greater than the other, then find what per cent this difference is of the other number.

87 is what per cent greater than 75?

87—75=12.

```
      .16
    -----
 75)12.00
     7 5
     7 5
    ----
     4 50
     4 50
    ----
```

$.16=16\%$ Ans.

478 is what per cent greater than 58?

$478-58=420$;

```
       7.24 4/29
    ----------
 58)420.00
    406
    ---
     14 0
     11 6
     ----
      2 40
      2 32
      ----
         8
```

$\frac{8}{58}=\frac{4}{29}$.

$7.24\frac{4}{29}=724\frac{4}{29}\%$ Ans.

$\frac{3}{4}$ is what per cent greater than $\frac{5}{9}$? $\frac{3}{4}-\frac{5}{9}=\frac{7}{36}$;

$\frac{7}{36}\div\frac{5}{9}=\frac{7}{20}=35\%$ Ans.

Find what per cent:

1. 49 is greater than 40?
2. 88 is greater than 30?
3. 126 is greater than 125?
4. $\frac{5}{9}$ is greater than $\frac{3}{7}$?
5. $4\frac{4}{5}$ is greater than $3\frac{3}{4}$?
6. 76.4 is greater than 7.64?

(13). **To find what per cent one number is less than another?**

This is similar to (9) in fractional parts. First find how much it is less than the other number, and then divide this difference by that number.

45 is what per cent less than 60?

60—45=15

```
      .2 5
    ------
 6$\not{0}$)1'5.0
```

$.25=25\%$ Ans.

2.15 is what per cent less than 3.6?

$3.6 - 2.15 = 1.45$

```
        .40 5/18
      ________
 3.6)1.4'50
      1 4 4
     ______
         10
```

$\frac{10}{36} = \frac{5}{18}$

$.40\frac{5}{18} = 40\frac{5}{18}\%$ Ans.

Find what per cent:

1. 147 is less than 250?
2. 265 is less than 625?
3. .0475 is less than .475?
4. $3\frac{3}{5}$ is less than $6\frac{1}{4}$?
5. $\frac{3}{8}$ is less than $\frac{2}{3}$?
6. $.6\frac{2}{7}$ is less than $4\frac{3}{7}$?

In solving concrete problems the per cent should always be connected with some object which is the base, or 100 per cent. For example, in problem "3" below Mary's age is the base.

Solution.

Let 100 per cent = Mary's age.

Then 125 per cent = John's age and 225 per cent = the sum of their ages.

225 per cent of Mary's age equal 36 years.

Hence Mary's age = 16 years. and John's age = 20 years.

Per cents cannot be added or subtracted in the concrete unless they are per cents of the same thing. 4 per cent of a yard, 5 per cent of a foot, and 6 per cent of an inch can be added only after they have been changed to per cents of the same unit.

4 per cent yd. plus 5 per cent ft. plus 6 per cent in. equal 12 per cent ft. plus 5 per cent ft. plus .5 per cent ft. equal 17.5 per cent ft.

It is best to state the preliminary work in concrete problems fully and explicitly. The solution of problem "5" below is here given as an example.

Solution.

Let 100 per cent = William's share.

Then 120 per cent = Samuel's share

and 180 per cent = Homer's share.

400 per cent = the sum of shares.

400 per cent of William's share = $400.

Hence William's share = $100.

Samuel's share = $120.

Homer's share = $180.

PROBLEMS.

1. A number which is 20% greater than 180 is 25% less than what number? Ans., 288.

2. The number which is 25% less than 180 is 25% greater than what number? Ans., 108.

3. John is 25% older than Mary, and the sum of their ages is 36 years. How old is each? Ans., M. 16; J. 20.

4. Mary is 40% younger than John and the sum of their ages is 40 years. How old is each? Ans., M. 15; J. 25.

5. Samuel has 20% more money than William, and Homer has 50% more than Samuel. They have together $400. How much has each? Ans., W. $100; S. $120; H. $180.

6. Mary has 20% more chickens than Susan, and Helen has $33\frac{1}{3}\%$ fewer than Mary. They have together 360 chickens. How many has each? Ans., M. 144; S. 120; H. 96.

7. Mr. Macy's 1909 crop of prunes was 20% less than his 1908 crop, and his 1910 crop was 50% more than his 1909 crop. His 1910 crop exceeded his 1908 crop by 6 tons. How much was his 1910 crop? Ans., 36 tons.

8. Mr. Bishop's fruit crop was 20% less than Mr. King's, and Mr. Ball's crop was 50% more than Mr. Bishop's. Mr. Ball had 7 tons more than Mr. King. How much had each? Ans., K. 35 tons; Bi. 28 tons; Ba. 42 tons.

9. If cloth loses 10% of its length in washing and dyeing, how much unwashed and undyed cloth is required to make 360 yards of cloth after it is washed and dyed? Ans., 400 yards.

10. Mr. Wise bought clothing at $16 a suit, and wishes to sell it at a gain of 15%. How should he mark it so that he may reduce the price 8% and still gain the 15%? Ans., $20.

11. A tree increased its height 50% the first year, was then cut back 20%, and the second year increased its height 33⅓%, when its full height was 8 feet. How high was it at first? Ans., 5 feet.

12. Mr. Slocum increased his weight 20%, then lost 25%, then gained 50%. He now weighs 216 lbs. How much did he formerly weigh? Ans., 160 lbs.

13. Mr. Schley sold his cow for $43.20. He had asked 50% more than the cost, and sold for 10% less than the asking price. What was his gain? Ans., Gain, $11.20.

14. If cloth shrinks 12½% of its length in washing and dyeing, what is gained in selling 840 yards of shrunken cloth bought unshrunken at 30c a yard, and sold at 40c, the cost of washing and dyeing being $4.50? Ans., $43.50.

15. Mr. Hale marked his suits at 20% above cost, and sold them at a discount of 10% at $14.85 each. What was his profit on 250 suits? Ans., $275.

16. Lean hogs are bought at 8c a pound, and fat rogs are sold at 9c, and it costs 5c for every pound a hog increases in weight. Mr. Crandall bought 150 lean hogs averaging 200 pounds each, and increased their weight 25%, then sold them. How much did he make? Ans., $600.

17. If cloth shrinks 12% of its length in washing and dyeing, how many yards of unshrunken cloth are required to make 22 suits, each suit requiring 12 yards of cloth after it is washed and dyed? Ans., 300 yds.

18. Mr. Hardy raised 29,280 lbs. of prunes in 1910. His 1910 crop was 20% less than his 1909 crop, and the crop of 1909 was 20% more than that of 1908. How much was his 1908 crop? Ans., 30,500 lbs.?

19. Dried peaches gain 10% in weight in processing. Find the profit on a shipment of 22 carloads of 15 tons each of processed fruit sold at 9c a pound; the unprocessed fruit was bought at 7c a pound, and the cost of packing and shipping was $250 per car. Commission at 4% was paid both for buying and selling. Ans., $7,844.

20. A cubic centimeter of water weighs one gram. Gold is 19.3 and silver 10.5 times as heavy as water. What is the weight of 48 cubic centimeters of an alloy of gold and silver of which 75% is gold? Ans., 820.8 gr.

21. Sea water is 2.6% heavier than fresh water. How much will the alloy mentioned in problem 20 weigh if suspended in sea water? Ans., 771.552 gr.

22. Mr. Thompson's fruit crop in 1911 is 80% of his crop in 1910, the price is 50% higher, and the expense of handling 10% higher. He had 35 tons in 1910, for which he received $75 a ton and paid $5 a ton for handling. What will his 1911 crop net him? Ans., $2,996.

23. An aviator traveled a certain rate the first hour, increased his speed 20% the second hour, decreased it 20% the third hour, and increased it 20% the fourth hour. He traveled 57.6 miles the fourth hour. How far did he travel? Ans., 215.6 mi.

24. It is 28 miles from San Jose to Mount Hamilton. A carriage travels 75% faster coming down than going up. A carriage leaves San Jose at 11 o'clock, remains at the

summit 3 hours, and returns at 1 a. m. What are the rates of travel up and back? Ans., 4 mi.; 7 mi.

25. The hind wheel is 15% larger than the fore wheel on a carriage. How many revolutions will each make while the fore wheel is gaining 45 revolutions? Ans., 300; 345.

26. Mr. Barnes sold his farm for $12,340. He had asked 25% more than the farm cost, and sold at a reduction of 12% on the asking price. What was the gain? Ans., $1,121.82.

27. Mr. Wells bought a house and lot for $4,800. He placed it on sale at 50% above cost, sold at a reduction of 20% on the asking price, and paid 5% of the selling price to the agent. What was the gain? Ans., $672.

28. Mr. Wilson purchased 40 A. of land at $75 an acre, and bought 3,600 orange trees at 60c each. 5% of the trees are found to be worthless, and 20% of the remainder died. He paid 25c each for having the trees set out, and $5 an acre each year for cultivation. What was the cost per living tree, including the land, in four years? Ans., $2.49.

29. Mr. Curtner sold his crop of prunes in 1912 for $4,704. He had 40 tons of French prunes. His crop of Silver prunes was 20% smaller and the price per ton 20% higher than the French prunes. At what price per ton did he sell each? Ans., F. $60; S. $72.

30. Mr. Jackson's live stock is valued at $3,200. Forty per cent of his animals are goats, 35% sheep, and the remainder hogs. A goat is worth ¾ as much as a sheep, and a sheep is worth ⅗ as much as a hog. Find the value of each kind of stock. Ans., $900 g.; $1,050 sh.; $1,250 h.

31. A hall committee paid $940.80 for 672 yards of carpet. It bought 12 per cent more carpet than was needed and the price paid per yard was 12 per cent higher than it should have been. How much should the carpet have cost? Ans. $750.

32. Green peaches lose 85 per cent of their weight in drying, and dried peaches gain 10 per cent in weight in preparation for packing. It costs $5 per green ton for picking and drying peaches and $2.40 per ton of packed fruit for packing. What are the net proceeds of 57¾ tons of packed fruit sold at 7c a pound? Ans. $6196.40.

33. Twenty per cent of an army were killed in battle, 30 per cent of the remainder died of wounds. The number which died of wounds exceeded the number killed in battle, 472. How many were left in the army? Ans. 6608.

34. If cloth shrinks one-ninth of its length in washing and dyeing, what is the gain in selling 720 yards of dyed cloth, bought undyed at 30 cts. a yard and sold after being dyed at 40 cts. a yard, if the cost of washing and dyeing is $4.50?
Ans., $40.50.

35. The outer walls of a building contain 5040 square feet, and each strip of rustic overlaps one-eighth of the width of another. What will be the cost of the rustic and painting for the building, if the rustic costs $35 a thousand and the painting costs 30 cts. a square yard?
Ans., $369.60.

36. If six pounds of green coffee make five pounds of dried, and green coffee is bought at 22 cts a pound and roasted coffee is sold at 30 cts. a pound, how much is gained by selling 1200 pounds of roasted coffee if the cost of roasting that amount is $2.75? Ans., $40.45.

BUSINESS CUSTOMS.

The application of percentage to different lines of business should be presented mainly from the information standpoint. The pupil will be

interested in an application only when he sees that it in some way touches the neighborhood interests. While he should be encouraged to find out for himself the prevailing business customs, the teacher should see that these customs are fully and clearly stated, and that they are learned.

Loss and Gain.

1. Loss or gain is reckoned on the cost.

2. Cost is 100 per cent for reckoning loss or gain.

3. Selling price is more than 100 per cent when there is gain.

3a. Selling price is less than 100 per cent when there is loss.

The pupil should assist in making statements 2 and 3.

Trade Discount.

1. The First Discount is reckoned on the list price; the Second Discount is reckoned on the first proceeds, and so on.

2. The List Price is 100 per cent for reckoning the first discount.

3. The First Proceeds is less than 100 per cent for first discount. It is 100 per cent for reckoning the second discount, etc.

Commission in Selling.

1. Commission is reckoned on the selling price.

2. The Selling price is 100 per cent for reckoning commission.

3. The Proceeds is less than 100 per cent.

Collections and similar transactions are on the same basis as selling on commission.

Commission in Buying.

1. Commission is reckoned on the purchase price. (That paid by the agent.)

2. The purchase price is 100 per cent for reckoning commission.

3. The entire cost is more than 100 per cent.

Property Insurance.

When this topic is taken up there should be a general discussion of the subject, as regards value, face of policy, premium, risk, etc.

Fire insurance is usually quoted as so many cents a year on the hundred dollars. The three year rate is double the one year rate, the five year rate is three times annual rate. The agent usually receives 15 per cent commission on the amount of premiums collected, and all the policy fee when one is charged. The premium is reckoned on the face value of the policy.

Life Insurance.

Discuss fraternal insurance and insurance companies. Also accident policies, endowment policies, straight life policies, who may and who may not be insured, etc.

The premium is usualy reckoned as so many dollars per thousand on the face value of the policy, payable annually, quarterly, or monthly as the case may be.

Taxes and Duties.

Discuss import duties, internal revenue, poll tax, property tax, their purpose, aim, and manner of collection.

The facts concerning property taxes in California are as follows:

1. Between the first Monday in March and the last day of June of each year, each property owner must furnish the assessor with a list of the property owned by him at noon on the first Monday of March.

2. During the month of July the County Board of Supervisors sits as a Board of Equalization. It examines the valuations made by the assessor and his deputies and raises

or lowers any valuations as it may think proper. The property owner has the right to go before the Board and ask that his assessment be lowered, or show cause why it should not be raised.

3. In August the State Board of Equalization examines the assessments of the counties, and raises or lowers any county assessment as it may think proper. Each county has a right to be heard thru its supervisors.

The State Board also places a valuation on the railway property of the State, and apportions this valuation among the counties in proportion to the number of miles of track in each county.

It also fixes the State tax rate.

4. In September the Board of Supervisors fixes the county and city tax rates.

5. The Auditor calculates the tax of each individual and must have his work completed by the second Monday in October.

6. Taxes are payable to the Tax Collector in two installments. The first installment consists of all the tax on personal property, and half the tax on real estate. It is delinquent if not paid on or before the last Monday in November, and a penalty of 15 per cent is added.

The second installment is half the tax on real estate and is delinquent if not paid on or before the last Monday in April and 5 per cent is added to all taxes remaining unpaid. If not paid before the delinquent tax list is published a charge of 50 cents is added for each piece of property delinquent. If not paid the property is sold to the State.

The teacher should show the pupils an assessment blank and a tax receipt, also a delinquent tax list.

In solving problems in loss and gain, commission, etc., the work should be written in good business form. Business blanks, bill heads, and ruled journal and ledger paper should be used when practicable. The required multiplications and divisions should be performed as side work, and in the additions and subtractions the decimal points should be kept in the same vertical column, and when there are several items oi debits and credits double columns should be used.

A few models are here given.

1. Mr. Copeland marked his suits 40 per cent above cost and sold them at 10 per cent discount on the marked price, receiving $24.57 each. How much was his profit on 50 suits?

Let 100 per cent equal cost of one suit.

Then 140 per cent equals marked price.

14 per cent equals reduction.

126 per cent equals selling price.

126 per cent of cost equals $24.57
cost of one suit equals $19.50

gain on suit equals ......$ 5.07
gain on 50 suits equals....$253.50

Side work.

```
         $19.50
       ---------
1.26)24.57' 00
     12.6
     ----
     1197
     1134
     ----
      630
      630
      $5.07
         50
     ------
    $253.50
```

2. Miss Wise bought a lot for $840 and built a house costing $2800. She rents the house at $35 a month and pays $1.60 a month for water, $55 a year for taxes and insurance on ¾ of the cost of the house at an annual rate of 45c. What per cent does her investment net her?

Solution.

Receipts.

Rent for 12 months $420

Expenses.

Water for 12 months	$19.20	
Taxes	55.00	
Insurance	9.45	83.65
Net receipts		$336.35
Cost of property		$3640
Rate of interest		9.2 per cent

Side work.

```
          .092
     ---------
3640)336.350
     327 6
     -----
       8 75
       7 28
       ----
       1 47
```

3. Mr. Harmon is assessed $12400 on real estate and $950 on personal property. The tax rate is $1.76. He pays all his taxes May 2nd. How much does he pay?

½ real estate tax	$109.12
Personal property	16.72
First Installment	125.84
Second Installment	109.12
Penalty first installment ..	25.168
Penalty second installment.	5.456
Amt. paid................	$265.58

4. A commission merchant sold on commission of 5 per cent 500 sacks, 55,000 lbs. of potatoes @ $1.20 per C.; 600 melons @ $9.25 per C. He paid $45 freight and $8.50 drayage. Find the amount due the consignor.

55000 lb. potatoes @ 1.20 per C...	$660.00	
600 melons @ 9.25 per C...	55.50	$715.50
Commission at 5%	35.78	
Freight	45...	
Drayage	8.50	89.28
Balance..............		$626.22

Problems.

1. Mr. Barnes sold his farm for $12,340. He had asked 25% more than the farm cost, and sold at a reduction of 12% on the asking price. What was the gain? Ans., $1,121.82.

2. Mr. Wells bought a house and lot for $4,800. He placed it on sale at 50% above cost, sold it at a reduction of 20% on the asking price, and paid 5% of the selling price to the agent. What was the gain? Ans., $672.

3. Mr. Johnson bought 576 sacks of potatoes at $1.20 a sack. 12½% of them spoiled. At what price per sack must he sell the remainder to realize a profit of 16⅔% on the whole investment? Ans., $1.60.

4. Mr. Johnson bought 576 sacks of potatoes at $1.20 per sack. 16⅔% of them had to be sold at a loss of 25%. At what price per sack must the remainder be sold to realize a profit of 20% on the whole investment? Ans., $1.548.

5. Mr. Conkling bought 500 boxes of oranges at $1.20 per box. 40% being large, were sold at a profit of 12½%. At how much per box must the rest be sold to realize a profit of 25% on the whole investment? Ans., $1.60.

6. Mr. Stockton and Mr. Thompson each agreed to sell 7,038 sacks of grain. It was found that Mr. Stockton had underestimated his crop 20% and Mr. Thompson had overestimated his crop 20%. How much has each? Ans., S. 8,797.5; T. 5,865.

7. Mr. A purchased a mower listat $160, at 30 and 15 off. He paid an agent 5% for making the purchase, and freight of $6.25. He sold the machine at 10% off the list price. What was the gain? Ans., $37.79.

8. Mr. B sold a house for $7,500, which was 25% more than it cost. He paid an agent 5% for making the sale. With the proceeds he bought another, which was afterward sold at a loss of 10%, no commission. What was the net loss or gain? Ans., $412.50.

9. Mr. Hale bought a house for $4,200, and spent $300 for repairs. He offered it for sale at $7,000, and aferward sold it at a reduction of 10%, and paid his agent 5% commission. How much did Mr. Hale make? Ans., $1,485.

10. Mrs. Prim bought a house for $3,600 and spent $200 in alterations and repairs. She pays a yearly tax at $2.50 on an assessed valuation of $2,250, insurance on the three years' plan on $2,500 at the annual rate of $.45, and a monthly water rate of $1.80. If the house is kept rented at $35 a month, what per cent does the investment net her? Ans., 8.8%.

11. Mr. Clark bought a store at $8,000 which he rents at $100 a month. He pays insurance at $1.10 on $6,000, taxes at $2.55 on $4,800, and estimates a yearly repair bill of $100. What per cent does the investment net him? Ans., 11.4%.

12. Mr. Martin bought a horse for $125, offered it for sale at 40% above cost, and sold it at a reduction of 10%. He paid an agent 5% for making the sale, and $12.50 for other expenses. How much did he gain? Ans., $12.13.

13. Mrs. Dawson offered her house for sale at 25% above cost, and afterward sold it at a reduction of 12%. The agent got 5% commission, and she received $4,723.40. What did she make or lose? Ans., $203.40 gain.

14. Mr. Buell bought a carriage listed at $500, receiving discounts of 25 and 10 off and paying an agent 5% for making the purchase and $12.50 freight. He sold the carriage at 15% discount on the list price, paying an agent 4%. What was his gain? Ans., $41.12.

15. A house was kept insured on the three years' plan for $2,500 at an annual rate of 45c. It burned the eleventh year. Find the cost of insurance, including a policy fee of $1 for each issuance of his policy. Ans., $94.

16. How much did the agent make and how much did the company receive from the transactions in problem fifteen? Ans., A. $17.50; Co. $76.50.

17. Mr. Kirk bought a lot for $800 and built a house costing $2,400, insures his house for $2,000 at 50c, pays taxes at $2.60 on an assessed valuation of $1,920. For how much per month must he rent the house to realize 8% net on his money? Ans., $26.33.

18. Mr. A bought a piano listed at $800 at 40 and 20 off. He paid an agent 10% for purchasing, and paid $45 for freight. He sold it at 15% discount. What was the gain? Ans., $212.60.

19. Mr. Christie bought a piano listed at $700 at 25 and 15 off. He afterward sold it to Mrs. Monroe, receiving $50 cash and a monthly payment of $6. After making seven payments Mrs. Monroe returned the instrument. Mr. Christie spent $5 for repairs and then sold the piano for $550. What was his total profit? Ans., $190.75.

20. Mr. Sherman bought, through an agent, a carload, 20 tons, of wheat at $1.75 per cwt., and paid $15 freight. He sold the same at $1.95 per cwt. He paid 4% commission for buying, and 2½% for selling. What was his gain? Ans., $17.50.

21. A commission merchant bought for Mr. Madsen 20 doz. chairs listed $45 per doz., at discounts of 20 and 10, and sold the same at 15% discount, com. for buying 2%, com. for selling 4%, other expenses $8.75. What was Mr. Madsen's profit? Ans., $64.69.

22. Wiley B. Allen bought a piano listed at $600 at 30 and 10 off, and paid an agent 5% for making the purchase. He sold the same piano at 20% discount and paid his

clerk 4% for making the sale. How much did he make? Ans., $63.90.

23. Mrs. Jamison bought a house and lot for $3,000 and spent $500 for alterations and repairs. She insured the house for $2,800 on the three years' plan at a basis rate of 45c, and pays taxes at $2.45 on an assessed valuation of $2,250. The house rents at $40 per month, and the water rate is $2 a month. What per cent does the investment pay? Ans., 11.21%.

24. Mr. Strong keeps his house insured on the three years' plan for $3,560, the annual rate being 65c, policy fee $1. The house burns during the thirteenth year. What does the agent make, and how much does the company receive? Ans., A. $39.71; Co. $196.69.

25. Mrs. Phillips bought a house and lot for $2,400, keeps it insured on the three years' plan for $1,600, at the annual rate of 45c, no policy fee. She pays taxes at $2.45 on $1,500, water at $1.60 per month, and rents the house for $25 a month. The house is vacant two months each year. What per cent does she realize on her money? (No water tax when the house is vacant.) Ans., $8\frac{3}{60}$%.

26. Mr. Mason bought a house and lot for $3,800 and made alterations costing $700. He placed it on sale for $6,000, afterward reduced the price 10% and paid the agent 5%. What per cent does he make on his investment? Ans., 14%.

27. The Home Union bought flour listed at $4.75 a bbl. at 10% discount, paid freight at 25c per bbl., and sold the consignment at 5% above list price. How much was made on 1,200 bbls.? Ans., $555.

28. Miss Thrifty bought a lot for $1,200, built a house for $3,200, insured it for three years on ¾ of value at 40c annual rate. She rented it at $35 a month, reserved $25 a year for repairs. She paid taxes at $2.55 regular and 35c special, on a valuation of $2,500, and pays water rate of $1.50 a month. Find per cent on investment. Ans., 6.775%.

29. Mr. Ross bought a house for $3,000 and spent $200 for repairs. He offered it for sale at $4,000, reduced his price 10%, and paid an agent 5% for selling it. What was his gain? Ans., $220.

30. Mr. Lion sells furniture on credit at an advancement of 25% on the cost. He collects through an agent 90% of the sales. The agent keeps 4% commission, and pays in $43,200. What is Mr. Lion's profit, and what per cent does he make on his goods? Ans., $3,200; 8%.

31. A contractor builds a house for $3,300, realizing a profit of 20% on the cost. The cost of the labor was to the cost of the material as 2 is to 3. Find cost of labor and material. Ans., L. $1,100; M. $1,650.

32. If wages should advance 20%, and material decline 20% in value, and the same house be built at the same price, what per cent would the contractor make on the cost? Ans., 25%.

33. Mr. Harper's property is assessed as follows: real estate, $6,500, personal property, $1,500. The regular rate is $2.45, the special rate 25c. Find each installment of his taxes. Ans., $128.25, $87.75.

34. How much will Mr. Harper's taxes be if they are all paid January 10th? Ans., $235.24.

35. Mrs. Hooker is assessed $4,500 on real estate and $500 on personal property. The rates are $2.45 regular and 25c special. She pays her taxes May 1st. How much does she pay? Ans., $152.89.

36. Mr. Bennett's house was assessed at $2,400, and his personal property for $200. The regular rate was $2.10, and the special rate 15c. He paid his first installment February 1st, and his second installment

May 2nd. What did his taxes cost him? Ans., $64.58.

37. Mr. Foley is assessed $12,400 on real estate and $2,600 on personal property. The regular rate is $1.55, and the special rate 15c. Mr. F. pays the first installment April 1st, and the second installment May 1st. How much do his taxes cost him? Ans., $282.71.

38. Mrs. Anderson has real estate assessed at $6,400 and personal property assessed at $3,640. The regular state and county rate is $1.65, and there is a special rate of 15c. Find each installment of her taxes. Ans., $123.12; $57.60.

39. What will her taxes be if paid March 1st? Ans., $199.19.

40. What will her taxes cost if all are paid May 1st? Ans., $208.22.

41. Mr. Reynolds sold a house at a loss of 25 per cent. He invested the money received in another house which he afterward sold for $4,104, gaining 20 per cent on its cost. What was his net loss. Ans., $456.

42. Mr. Conkling sold 40 per cent of a carload of potatoes at a profit of 50 per cent, 25 per cent at a profit of 20 per cent and the remainder at a loss of 33⅓ per cent. He received altogether $408. What was his gain? Ans., $48.

43. Mr. Anderson bought a piano listed at $600, with discounts of 40 and 20 off. He sold it at a discount of 25 per cent and paid an agent 10 per cent commission for selling. What was Mr. Anderson's gain, and what was the agent's commission?
Ans., $117.

44. Mr. Buell bought a carriage listed at $500, receiving discounts of 25 and 10 off, paying an agent 5 per cent for making the purchase, and $12.50 freight. He sold the carriage at 15 per cent discount. What was his gain? Ans., $59.125.

45. Mr. Cowper received $114 as the proceeds of the sale of a mower. He had allowed a discount of 4 per cent and had paid an agent 5 per cent commission for making the sale. The mower had been purchased at 30 per cent discount. What was Mr. Cowper's gain? Ans., $26.50.

INTEREST.

Interest is usually charged at a certain rate per cent per annum.

The time is found by counting from one date to another, ordinarily by compound subtraction, sometimes however, by finding the actual number of days. When compound subtraction is used, 30 days are called a month and 12 months, or 360 days, a year. When the time is found by counting the actual number of days 360 days are called a year in ordinary commercial transactions, and for exact interest 365 days make a year. Exact interest is reckoned by the large city banks only, and by the United States government.

When interest is to be calculated for years or years and months, it is only necessary to find the interest for one year and then multiply this by the number of years and fraction of a year.

Find the interest on $265.40 for 3 years 9 months at 5 per cent.

Solution.

3 yr. 9 mo. equal 3¾ years.

$$\begin{array}{rl}
\$264.40 & \\
.05 & \\
\hline
\$13.22\cancel{00} & \text{Int. for 1 yr.} \\
3\tfrac{3}{4} & \\
\hline
9915 & \\
3966 & \\
\hline
\$49.58 & \text{Int. } 3\tfrac{3}{4} \text{ yr.}
\end{array}$$

When interest is to be found for days or months and days any one of several different methods may be used.

Cancellation Method.

Reduce the time to a fraction of

a year, indicate the work and use cancellation.

Find the interest on \$375.60 for 7 months 18 days at 5 per cent.

Solution.

7 mo. 18 da. equal $^{19}\!/_{30}$ yr.

$$\overset{.1252}{\cancel{375.60}} \times \frac{5}{1\cancel{00}} \times \frac{19}{\cancel{30}}$$

$\$.1252 \times 5 \times 19 = \$11.894$ Ans.

To apply the method skillfully requires that attention be given to three things:—

1st—Reducing the time to a fraction of a year.

2nd—Indicating the work.

3rd—The canceling.

First. It is best as a rule to change the days to a fraction of a month, unite the result with the months, then change to a fraction of a year. When days only are given, write 360ths of a year and reduce to its lowest terms. Thus,

6 mo. 18 da. $= 6\frac{3}{5}$ mo. $= \frac{11}{20}$ yr.

4 mo. 8 da. $= 4\frac{4}{15}$ mo. $= \frac{16}{45}$ yr.

7 mo. 20 da. $= 7\frac{2}{3}$ mo. $= \frac{23}{36}$ yr.

4 mo. 15 da. $= 4\frac{1}{2}$ mo. $= \frac{3}{8}$ yr.

2 yr. 4 mo. 15 da. $= 2\frac{3}{8}$ yr.

Reduce to fractions of a year 3 mo. 18 da., 5 mo. 12 da., 2 yr. 4 mo. 9 da., 1 yr. 7 mo. 10 da.

It should be noted that the factors of 30 are 2-3-5; those of 12, 2-2-3. If the number of days does not contain a factor 2, 3, or 5, it is best to reduce the months and days to days and place the result over 360.

When exact interest is required find the actual number of days and place over 365.

Second—Write the rate as a common fraction and express the time as a proper or improper fraction as the case may require. See examples solved above.

Third—It is best, as a rule, to cancel the ciphers in the denominators first and place a mark in the dollars as many places to the left of the decimal point as there are ciphers in the denominators which have been canceled. Do not cancel a cipher to the right of the decimal point, for this makes no change in the value of the number. When other factors have been canceled into the dollars, the mark should be replaced by the decimal point.

When the denominator has been reduced to a number not greater than 12 so that short division may be used, little is gained by further cancellation. It is as easy to divide by 9 as by 3, by 8 as by 4.

Six Per Cent Method.

First get the interest on \$1 for the given time at 6 per cent.

The interest on \$1 for 1 year is \$.06.

The interest on \$1 for 2 months is .01.

The interest on \$1 for 1 month is .005.

The interest on \$1 for 6 days is .001.

The interest on \$1 for 1 day is $.000\frac{1}{6}$.

Hence the rule:—

To get the interest on \$1 for any time at 6 per cent. Multiply .06 by the number of years. Divide the number of months by 2, call the quotient cents and the remainder if any 5 mills. Divide the number of days by 6, call the quotient mills and the remainder 6ths of a mill. The sum of the results is the required interest.

Find the interest on \$1 for 3 yr. 7 mo. 19 da. at 6 per cent.

\$1 @ 6 per ct. for 3 yr	$= .18$
\$1 @ 6 per ct. for 7 mo.	$= .035$
\$1 @ 6 per ct. for 19 da.	$= .003\frac{1}{6}$
Total	$= .218\frac{1}{6}$

With a little practice the result may be found by inspection. Follow the order given above, and add results as you proceed. Thus: .18, .215, $.218\frac{1}{6}$.

For 4 yr. 9 mo. 24 da. The results are thought out as follows:

.24, .285, .289.

Find the interest on $1 at 6 per cent for

2 yr. 8 mo. 18 da.

5 yr. 6 mo. 12 da.

7 yr. 7 mo. 15 da.

1 yr. 9 mo. 11 da. etc.

Second. To find the interest on any principal for any time at 6 per cent. Multiply the principal by the interest on $1 at 6 per cent for the given time.

Find the interest on $247.40 for 3 yr. 10 mo. 18 da.

Interest on $1 for the given time is .233.

```
$247.40  Prin.
    .233  Int. on $1.
---------
   74220
  74220
 49480
---------
$57.64420 Ans.
```

Third. To find the interest on any principal at any rate. Find the interest at 6 per cent and increase or decrease the result by such a fraction of itself as the per cent is greater or less than 6.

Use of Interest Tables.

Books of tables are prepared giving the time from any date to any other in the year, and data from which the interest on any sum at any ordinary rate for any desired number of years, months and days may be obtained by addition. These books are used largely by bankers and others who have much interest calculating to do.

Solve the first four problems given below by cancellation, the next four by the six per cent method, the next by either method, and the last two by using the actual number of days and 365 days to the year.

Find the interest:

1. On $485.40 at 7 per cent from Jan. 10 to July 25, 1914.
2. On $296.52 at 4½ per cent from Dec. 16, 1913, to Aug. 4, 1914.
3. On $956 at 8½ per cent from Sept. 10, 1914, to Jan. 26, 1915.
4. On $76.85 at 9 per cent from Feb. 13, 1915, to May 17, 1916.
5. On $283.56 at 6 per cent from Oct. 8, 1914, to April 26, 1916.
6. On $4967.25 at 6 per cent from May 7, 1913, to Jan. 16, 1916.
7. On $274.45 at 5 per cent from Nov. 6, 1913, to June 1, 1914.
8. On $865.27 at 7½ per cent from Oct. 18, 1914, to Feb. 28, 1916.
9. On $1360.32 at 5½ per cent from May 21, to Dec. 14, 1915.
10. On $675.80 at 8 per cent from April 17, 1914, to July 24, 1915.
11. On $674.82 at 6 per cent for 3 yr. 7 mo. 11 da.
12. On $206.15 at 4 per cent for 7 yr. 3 mo. 18 da.
13. On $267.83 at 6 per cent from June 27 to Sept. 23, 1914.
14. On $7658.25 at 5 per cent from Feb. 18 to May 27, 1916.

COMPOUND INTEREST.

When interest is made payable at stated intervals, if it is not paid when due, it is usually added to the principal and bears interest. In such cases interest is said to be compounded. Postal and other savings banks pay compound interest.

When interest is to be compounded the result for long terms is found by the use of compound interest tables.

When the use of a table is not convenient, the interest is added to the principal at the end of each term, and this becomes the principal for the succeeding term or part of a term as the case may be.

A good concise form saves time and is a safeguard against mistakes.

Find the compound interest on \$375 for 1 yr. 9 mo. 10 da. at 4 per cent compounded semiannually.

Solution.

4 per cent is 2 per cent for a half year or term.

1 yr. 9 mo. 10 da. = 3 terms + 3 mo. 10 da.

3 mo. 10 da. = $\frac{5}{9}$ of a term

1st Prin.	\$375.	× .02
1st Int.	7.50	
2nd Prin	382.50	× .02
2nd Int.	7.65	
3rd Prin.	390.15	× .02
3rd Int.	7.803	
4th Prin.	397.953	× .02 × $\frac{5}{9}$
4th Int.	4.422	
Amt.	\$402.375	
Prin.	\$375.	
Int.	\$ 27.375	

Side work:

$$\frac{39\cancel{7}.953 \times \cancel{2} \times \cancel{5}}{1\cancel{0}\cancel{0} \quad 9 \quad 1\cancel{0}} = \frac{39.7953}{9} = 4.422$$

1. Find the amount of \$2100 for 2 yr. 6 mo. at 6 per cent compounded semiannually. Ans. \$2434.48.

2. A note for \$1200 dated Nov. 7, 1912, with interest at 8 per cent compounded semi-annually was paid Jan. 1st, 1915. How much was due, no payment having been made. Ans., \$1420.69.

3. Henry Moss deposits \$100 in a savings bank Jan. 1 and July 1 of each year beginning Jan. 1, 1914. The bank allows 4 per cent interest compounded semi-annually. If the practise is continued and no money is withdrawn how much will be due him Dec. 31, 1916? Ans., \$643.43.

4. Mr. Allen has a school bond for \$1000 bearing 6 per cent interest payable semi-annually, the bond to be paid in five years. If Mr. Allen deposits his interest payments in a bank which pays interest at 4 per cent payable semi-annually, how much will he have to his credit at the end of the five years inclusive of the final payment? Ans., \$1328.47.

5. Miss Stoner opened an account with a savings bank Jan. 1, 1914, and deposits \$5 each Thursday. The bank pays 4 per cent compound interest allowing interest on the amount on deposit on the first day of any month and adds the interest July 1st and Jan. 1st. What will be due on her account Jan. 1, 1915? Ans., \$269.89.

6. Find the amount of \$376 for 1 yr. 8 mo. 10 da. at 6 per cent compounded quarterly. Ans., \$415.93.

Partial Payments.

It is not unusual for partial payments to be made on notes before the time for final settlement. The study of Partial Payments is therefore of value in itself, besides giving practice in computing interest, in the use of good forms and in keeping track of the work in a series of operations.

Before commencing the work the data should all be taken down as shown below, each date and its corresponding payment being written above the date and payment immediately preceding it.

Next, all the times should be found in order by compound subtraction and the corresponding payments brought down. Then the calculations should proceed step by step, care being taken that the interest calculations are clearly indicated, that in the addition and subtraction work the decimal points are kept in the same vertical line, and that each payment is checked as soon as it is used.

The United States rule is commonly followed when the note runs more than a year. When the time is a year or less the Mercantile rule is usually followed, tho there are no fixed customs for either rule.

United States Rule. Find the amount of the principal to a time when a payment or the sum of two or more payments equals or exceeds the interest due, and from the amount subtract such payment or payments.

With the remainder as a new principal proceed as before.

Model I.

(Each payment exceeds the interest due.)

A note for \$5000 was given Aug. 2, 1908, bearing interest at 6 per cent. The following payments were endorsed: Paid Sept. 9, 1908, \$500; May 12, 1909, \$350. What amount would settle the note July 2, 1909?

Yr.	Mo.	Da.	
1909 —	7 —	2	....
1909 —	5 —	12	\$350
1908 —	9 —	9	\$500
1908 —	8 —	2	....
$.006\frac{1}{6}$ —	1 —	7	\$500 ✓
.0405 —	8 —	3	\$350 ✓
$.00\frac{5}{6}$ —	1 —	20	.... ✓

$5000 \times .006\frac{1}{6} = \$\ 30.833$

$4530.83 \times .0403 = \$183.498$

$4364.33 \times .00\frac{5}{6} = \$\ 36.369$

Rate 6%

Prin.	500	..
Int.	30	83
Amt.	5030	83
1st Pay.	500	..
Bal.	4530	83
Int.	183	50
Amt.	4714	33
2nd Pay.	350	..
Bal.	4364	33
Int.	36	37
Amt.	4400	70

Model II.

(The interest exceeds a payment.)

Principal \$850. Date May 10, 1905. Rate 7 per cent. Endorsements:—July 15, 1906, \$130; June 1, 1907, \$46; Dec. 12, 1908, \$380. What was due May 10, 1909?

Yr.	Mo.	Da.	
1909—	5—	10	——
1908—	12—	12	\$380
1907—	6—	1	\$ 46
1906—	7—	15	\$130
1905—	5—	10	——
1—	2—	5	\$130 ✓
	—10—	16	\$ 46 ✓
1—	6—	11	\$380 ✓
	4—	28	——

$\frac{8'50 \times 7 \times 85}{1\cancel{0}\cancel{0}\ \ 72} = \$70.24$

$\frac{'790.24 \times 7 \times 79}{1\cancel{0}\cancel{0}\ \ 9\cancel{0}} = \$48.556$

$\frac{790.24 \times 7 \times 551}{1\cancel{0}\cancel{0}\ \ 26\cancel{0}} = \$84.657$

$\frac{'497.46 \times 7 \times 37}{1\cancel{0}\cancel{0}\ \ 9\cancel{0}} = \$14.315$

Rate 7%

Prin.	850	..
Int.	70	24
Amt.	920	24
1st Pay.	130	..
Bal.	790	24
Int.	48	56
Int.	84	66
Amt.	923	46
2nd & 3rd Pay.	426	..
Bal.	497	46
Int.	14	32
Amt.	511	78

1. A note for $2150, dated Mar. 10, 1913, and bearing interest at 8 per cent was indorsed as follows:

Sept 25, 1913, $275.
Mar. 10, 1914, $365.

How much was due Mar. 10, 1915? Ans., $1770.39.

2. A note for $965, dated Jan. 16, 1912, and bearing interest at 7 per cent has the following indorsements:

May 7, 1913, $150.
Jan. 16, 1914, $30.
June 28, 1914, $275.

How much was due Jan. 16, 1915? Ans., $620.64.

3. Mr. Rawlins gave a note for $1400 Oct. 12, 1912, with interest at 6 per cent payable semi-annually. He paid the interest and $250 on the principal at each interest payment. How much was each payment and how much remained due Oct. 12, 1914? Ans., $271, $267.25, $263.50, and $659.75.

Merchant's Rule. Find the amount of the principal from its date to the time of settlement.

Find the interest on each payment from the time it was made till the time of settlement.

From the amount of the principal subtract the amount of the payments and their interest.

Model.

Each date is subtracted from the last.

A note for $500 dated May 15, 1907, has the following endorsements: July 10, 1907, $145; Oct. 16, 1907, $175. How much was due Jan. 1, 1908, interest at 6 per cent?

Yr.	Mo.	Da.	
1908 —	1 —	1	....
1907 —	10 —	16	$175
1907 —	7 —	10	$145
1907 —	5 —	15	....
.037⅔ —	7 —	16	....√
.0285 —	5 —	21	$145√
.0125 —	2 —	15	$175√

$500 × .037⅔ = $18.833
$145 × .0285 = 4.132
$175 × .0125 = 2.187

Rate 6%

Prin.	500	..		
Int.	18	83	$518	83
1st Pay.	145	..		
Int.	4	13		
2nd Pay.	175	..		
Int.	2	19	326	32
Bal.			$192	51

In connection with this subject discuss notes, indorsements, life of note, etc.

1. A note for $975, dated April 16, 1913, and bearing interest at 6½ per cent was indorsed as follows:

June 20, 1913, $375.
Sept. 16, 1913, $450.

How much was due Mar. 16, 1914? Ans., $175.46.

2. On a note for $1200, dated July 1, 1913, bearing interest at 7 per cent payments of $300 were made at the end of each quarter. How much was due July 1, 1914? Ans., $352.50.

Bank Discount.

Commercial banks make loans for not to exceed six months, as a rule, with interest payable quarterly or semi-annually. The borrower receives the face of the note and pays the face and accrued interest. Some banks collect the interest in advance, but this is not customary in California.

Commercial paper issued by large manufacturing firms or packing companies and county or city warrants are discounted by deducting the interest on the face from the time the paper is purchased by the bank to the time it is due. Notes are seldom bought in this way.

If an interest-bearing note is discounted the discount is reckoned on the amount that will be due on the note at the time it falls due.

Find the discount by counting the actual number of days and using 360 days for a year. Days of grace are not allowed in California.

This interest on notes or other paper discounted is called **bank discount**, and the amount paid the holder is the **proceeds.**

Find the bank discount and proceeds of a warrant for $220 due in 75 days, and discounted at one per cent a month.

Solution.

Face	$220
Discount	5.50
Proceeds	$214.50

Side work.

75 da. equal $\frac{5}{2}$ mo.

2.20\times\frac{5}{2}$ equal $5.50

1. A time draft for $167.50 due in 4 months is discounted at 7 per cent. Find the discount and proceeds.

2. Mr. Anderson holds a 90 da. time draft for $1062.80, dated Aug. 15. He sells it Oct. 5 to the First Nations Bank which discounts it at 6 per cent. Find the proceeds.

3. Mr. Ryan holds a note for $975.60 due in 5 months and bearing 5 per cent interest. He sells it to the Commercial Bank at 8 per cent bank discount. Find the proceeds.

4. A note for $4760 dated July 1st, 1914, and bearing interest at 6 per cent and due 6 months after date was discounted Sept. 10 at 7 per cent. Find the proceeds.

It is sometimes required to find the face of a note which will yield a given amount when discounted at a bank, tho such a problem would seldom if ever arise in practise. To solve such a problem find the proceeds of one dollar and divide the given proceeds by it.

Find the face of a 90 day note which will give proceeds of $1200 when discounted at a bank at 8 per cent.

Solution.

Int. on $1 for 90 da. at 8 per cent equals .02.

$1.00—$.02 equal $.98.

```
           12 24.89
     ----------------
.98)$1200.00'00
     98
     ---
     220
     196
     ---
      240
      196
      ---
       440
       392
       ---
        880
        784
        ---
         96
```

Ans. $1224.49.

1. Mr. Watson has purchased an automobile for $1360. If the bank charges 7 per cent discount for how must a 60 day note be drawn in order to secure the purchase price?

2. Mr. Peart has bargained for an orchard for $22500. He has $18000 cash and is to pay the balance in four months. For how much must his note be drawn that the

balance may be realized if the note is discounted at a bank at 6½ per cent?

Present Worth.

If a sum of money is to be paid at a future date its present value would be the principal which would amount to that sum at the given time.

1. What is the present value of $800 due in 9 months without interest, money being worth 6 per cent?

One dollar amounts to $1.045 in 9 months. $800 divided by $1.045 will give the desired principal which is $765.55.

2. A bond of $1000 due in three years bears 6 per cent interest payable annually. If money is worth 5 per cent what is the present value of the bond?

There will be three payments, one of $60 in 1 year, another of $60 in two years, and a third of $1060 in three years. The present value of the bond is the sum of the present values of these three payments. Using compound interest which is proper when payments are made at stated intervals the answer is $1027.23.

3. Mr. Crittenden has his farm leased for five years at a cash rental of $5000 a year payable at the end of the year. What is the present worth of the lease reckoning money worth 6 per cent simple interest? Ans., $2196.97.

In the following problems use compound interest making use of a table.

4. If money is worth 4%, what should be paid for a bond for $1,000 payable in 5 years, and bearing 5% interest payable annually? Ans., $1,044.51.

5. Mr. Brown bought a house for $4,000. He is to pay for it in five equal annual payments, each payment to be made at the end of the year. Interest at 6% is allowed on unpaid balances. What is the annual payment? Ans., $949.58.

6. Mr. Clements proposes to place $100 in the bank to the credit of his son each birthday from the 16th to the 21st, inclusive. The bank allows 4% interest compounded semi-annually. How much will the son have to his credit when he is twenty-one? Ans., $663.96.

7. Mr. Moore, who is 60 years old, wishes to deposit sufficient money in a savings bank to meet the annual payment of $60 on his insurance policy for ten years, the payments to be made at the end of each year. The bank allows 4% interest compounded semi-annually. How much must Mr. Moore deposit? Ans., $485.68.

8. Mr. Simpson offers his farm for $5,000 cash, or $6,060 payable in three equal annual payments without interest, the payments to be made at the end of the year. If money is worth 6%, which proposition is the best, and how much? Ans., $5,000 is $346.02 better for the purchaser.

MENSURATION.

Areas and volumes of rectangular figures offer no serious difficulty and may be presented in an elementary way as early as the third year, thus furnishing excellent concrete material for the application of multiplication and division. Care should be taken that the pupil does not form the habit of saying and thinking that feet multiplied by feet give square feet, inches multiplied by inches give square inches, square feet multiplied by feet give cubic feet, square inches multiplied by inches give cubic inches, and so on. These notions arise from the fact that a rectangle 4 inches by 5 inches for example contains 4 times 5 or 20 square inches and a rectangular solid 3 inches by 4 inches by 5 inches for example contains 3 times 4 times 5 cubic inches.

It is best at first to associate mensuration work with the setting out of trees or vines in rows to form an orchard or vineyard, and the packing of eggs and fruit in boxes, of canned goods and other packages in cases, and similar work. A set of pasteboard forms used for keeping the eggs in a box separate will be very suggestive. It will be seen that each row will hold 6 eggs and that there are 6 rows, hence one form will hold 6 times 6 eggs, or 36 eggs. Each of these forms makes a layer in one end of an egg box and there are five layers in the box placed one above another. An egg box therefore contains 2 times 5 times 36 or 360 eggs. How many dozens? In like manner a case of canned corn contains 2 layers of cans placed in rows of 3 by 4.

Have pupils cut two differnt colors of card board into square inches. These may be built into rectangles of different shape. The pupils will soon be able to determine that a rectangle 5 inches by 6 inches for example will contain 5 times 6 or 30 square inches. They can also determine beforehand that it requires 21 square inches to make a rectangle 3 inches by 7 inches. How many square inches are required to make a square foot? How many square feet to make a square yard? Why? If square inches of cardboard are cut diagonally into triangles, many pleasing patterns may be made. Try such exercises with children of the third and fourth years and note the results.

The school should be supplied with cubic inch blocks of wood. These may be used in building up rectangular solids, and thru this work the pupil will learn how to determine the number of cubic inches in a solid of given dimensions. Do not teach rules for mensuration before the work is regularly taken up in the eighth year.

1. Draw a rectangle whose sides are respectively 4 in. and 5 in. Draw lines dividing it as shown in Fig. 39.

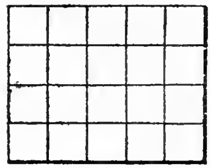


Fig 39

CHART 1.

Note that the figure is divided into squares one inch on a side, *i. e.*, square inches, that there are five squares in each horizontal row; that there are four such rows; that therefore the area is 4×5 sq. in., or 20 square inches.

In like manner show by diagram that

144 sq. in. = 1 sq. ft.
9 sq. ft. = 1 sq. yd.
$30\frac{1}{4}$ sq. yd. = 1 sq. rd.

To find the **area** of a given surface is to find the number of squares of a given kind that it contains.

From the problems given above, it will be seen that to find the area of a rectangle—**Multiply the number of units in the length by the number of units in the width.**

9. What is the area in sq. m. of a square whose side is 27 m.

10. Find the area in sq. cm. of the surface represented in fig. 40.

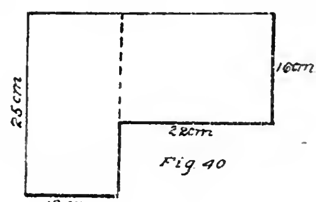


Fig 40

11. Find the area in sq. cm. of the surface represented in fig. 41.

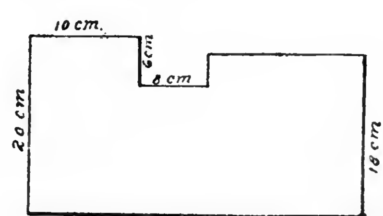


Fig 41

CARPETING.

Carpet is usually 3 ft., or 27 in. wide, and a yard of carpet means one yard in length without regard to the width.

Strips.—In finding how many yards of carpet are required it is necessary to decide which way the strips are to be laid and then to determine how many strips are required. A fractional part of a strip must be reckoned as a whole strip.

Matching.—Most carpet has a well defined pattern which should be matched along the edges of the strips. To do this it is necessary that each strip shall begin at the same point of the pattern. Hence, in determining the length of a strip full pattern lengths must be reckoned on all strips after the first.

Border.—Sometimes a carpet is surrounded with a border. This border is matched at the corners by beveling and its length must equal the distance around the room.

1. A room 13′ x 16′ is to be covered with carpet 27″ wide, the strips to run lengthwise. How many yards are required if there is no waste in matching?

2. How many yards of carpet one yard wide are required to cover a room 20′ x 22′, the strips running lengthwise, if 8″ are lost in matching all strips except the first?

3. A room 14′8″ x 16′6″ is to be covered with carpet 27″ wide at $1.40 a yard. The strips are to run lengthwise and there is a 12 inch pattern. How much will the carpet cost?

4. A room 16′ x 18′ is to be carpeted with carpet 27″ wide at $1.50 a yard surrounded with a border 16″ wide at $1.35 a yard. The carpet has a 15 inch pattern and the strips are to run lengthwise. Find the cost of the carpet and border.

Papering and Plastering.

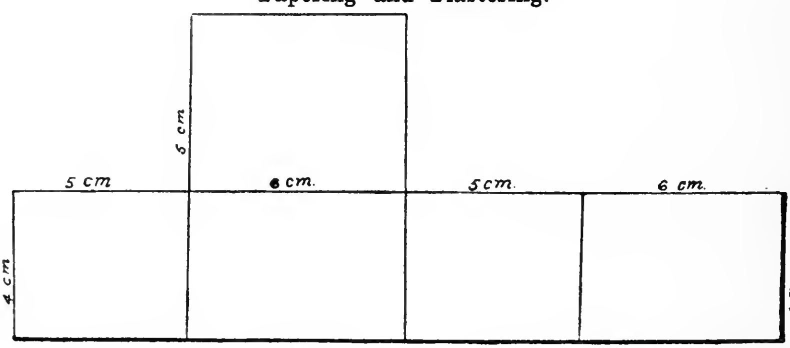


Fig 42

CHART II.

Take a box 2½ in. long, 2 in. wide and 1½ in. deep without top. Cut along the edges and spread out as shown in figure 42. This will represent the walls and ceiling of a room.

Paper is sold by the double roll sixteen yards long, or the single roll of eight yards, the width being 18 inches as a rule. Border is sold by the running yard.

There is no fixed rule for allowance for opening. Some contractors deduct one half the openings, others allow 2 square yards for each single opening. In practise paper-hangers take sufficient paper from the stores with the privilege of returning unused rolls.

Plastering is done by the square yard and the allowance for openings is the same as for papering.

1. Find the cost at $.25 a double roll for paper and border at 3c a yard for a room 18′ x 22′, 9′6″ high, there being 2 windows and 2 doors.

2. A house 24′ x 30′, with walls 9′ is divided into four rooms. There are ten windows, two outside, and five inside doors. Find the cost of plastering at 27 cents a square yard.

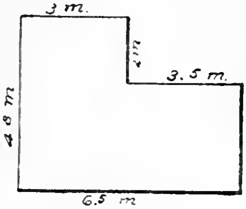


Fig. 43.

3. Figure 43 represents the floor plan of a room 20′ x 26′, the L ends each being 12′, height of ceiling 9′. There are six windows 3′6″ x 5′2″, and two doors 3′ x 6′8″. Find the cost of plastering the room at 30 cents a square yard allowing for half the openings.

4. Find the cost at $1.25 a yard of carpeting the room described in problem 3 with carpet a yard wide containing a 16 inch pattern, the strips running lengthwise.

Lumber Measure.

Lumber is sold by the thousand board feet except that moldings and the like are sold by the running foot.

A **board foot** is a square foot of lumber one inch thick. A thickness less than an inch is counted as an inch.

Thicknesses of more than an inch are reckoned in inches and quarters of an inch. When lumber is surfaced on one side, or sized, its thickness is reduced one eighth of an inch, but the full thickness is used in calculating the amount of lumber.

In tongue and grooved lumber, such as flooring, a 3 inch board covers 2½ inches of floor space, a 4 inch board covers 3½ inches. In calculating the amount of 3 inch flooring required for a building, find the amount of floor space and add ⅕ of it. For a 4 inch flooring add ⅐.

Rustic and other siding overlaps about an inch. Hence, for 8 inch rustic find the number of square feet in the walls to be covered and add ⅐.

In calculating the amount of lumber change the length and width to feet and the thickness to inches.

Find the cost of 36 pieces of redwood 2″ x 10″ by 32′ long @ $22 M.

Solution.

$$\frac{36\times2\times8}{3} = 192 \text{ board feet} =$$

.192 M.

$$\begin{array}{r} .192 \\ 22 \\ \hline 384 \\ 384 \\ \hline 4.224 \end{array}$$

Ans. $4.22.

1. Find the cost of the following bill of lumber at $21 M. Make out bill.

28 pieces 2″ x 12″, 34′ long.
48 pieces 1″ x 8″, 16′ long.
20 pieces 1½″x 10″, 14′ long.
16 pieces 1¾″x 6″, 18′ long.

2. Find the cost at $32 M. of 8 inch rustic to cover a building 36′ x 54′ outside measurement, the walls being 14′ high.

3. Find the cost at $38 M. of 3 inch flooring for a one story L shaped building 30′ x 36′ with L ends each 18′ wide.

4. Find the cost at 11c a foot for chalk molding and 6c a foot for upper molding for a room 33′ x 35′ with two doors each 4′ wide and six windows each 3′6″ wide.

Shingling.

Shingles are put up in bundles of 250 shingles each and sold by the thousand, only whole bunches being sold.

While practically shingles are of varying widths a shingle, commercially considered, is four inches wide. Shingles are usualy laid with 4 inches or 4½ inches exposed. When

laid 4 inches to the weather one shingle covers 16 square inches, or 9 shingles cover a square foot. When 4½ inches are exposed 8 shingles cover one square foot.

Find the cost at \$2.40 M. of shingles laid 4½ in. to the weather to cover a roof each side of which is 24′ x 60′.

Solution.

$2 \times 24 \times 60 \times 8$ equal 23040.

23040 shingles equal 23.04 M.

23.25 M. required.

$23.25 \times \$2.40$ equal \$55.80 Ans.

1. Find the cost at \$2.50 M. of shingles for a roof each side of which is 23′ x 41′ the shingles being laid 4″ to the weather.

Land Measure.

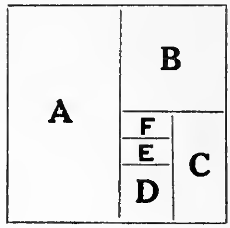
For a full description of land measure in the United States see the California Advanced Arithmetic, pages 297-303.

A township is six miles square and contains 36 sections of 640 acres each if full size.

The sections of a township are numbered as shown in the diagram.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

When land is surveyed by the government, posts or stones are placed along section and township lines every half mile, thus establishing section and quarter section corners. An interior quarter section corner must be located at the intersection of lines joining opposite quarter section corners. A section may be further subdivided and the portions described as shown below.



A. W. ½ of sec.

B. N.E. ¼ of sec.

C. E. ½ of S.E. ¼ of sec.

D. S.W. ¼ of S.E. ¼ of sec.

E. S. ½ of N.W. ¼ of S.E. ¼ of sec.

In determining the amount of land in a portion of a section deal with the squares as units bearing in mind that a section has 640 acres, a quarter section 160 acres and a quarter of a quarter 40 acres, etc. F in the diagram, for example, has ½ of 40 A. or 20 A.

In following examples locate the land in the section and the sections in the township.

1. Find the value of W. ½ of N.E. ¼ S.W. ¼ of sec. 26 at \$65 an acre.

2. Find at \$2.25 a rod the cost of fencing N.E. ¼ of N.W. ¼ of S.W. ¼ of sec. 17.

3. Mr. Wallace purchased the S.E. ¼ of sec. 21, the W. ½ of S.W. ¼ of sec. 22, and the N.W. ¼ of N.W. ¼ of sec. 27, all in T. 4 S., R. 2 W. at \$85 an acre. Find the cost of the land and the number of rods of fence required to enclose it

Rectangles, Squares.

In the figures in this chapter, use the same lengths for a, b, and c respectively thruout.

1. By the **square** on a line is meant the square of which that line is one side.

The square on the line, **a**, is the square which has **a** for one side, (Fig. 70). It is written a^2, and is read, The square on the line **a**.

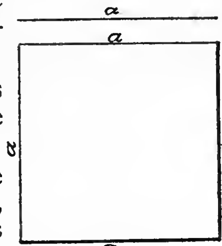


Fig. 70.

Fig. 71.

2. The **rectangle** of two lines means the rectangle whose length is one of the lines and whose width is the other.

The rectangle of the lines, a and b, is the rectangle whose length is **a** and whose width is **b**, (Fig. 71). It is written rect. ab. and is read The rectangle of the lines a and b, or Rectangle ab.

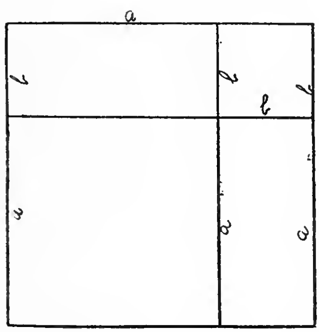


Fig. 72.

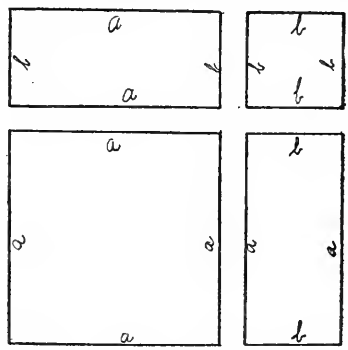


Fig. 73

$$(a+b)^2=a^2+b^2+2 \text{ rect. } ab$$

3. Find the sum of the lines, **a** *and* **b**, and construct a square on this sum, (Fig. 72). Divide it by drawing lines as shown in the figure. Cut this out and cut out another of exactly the same size and shape. Cut the second figure along the inside lines. Arrange the figures as shown in Figure 73.

The square on the sum of a and b equals the square on a, plus the square on b, plus twice the rectangle of a and b.

Building a Square.

1. Find the square of 84.

$$\begin{aligned} 84 &= 80+4 \\ 80^2 &= 6400 \\ 2\times80\times4 &= 640 \\ 4^2 &= 16 \\ \hline 84^2 &= 7056 \end{aligned}$$

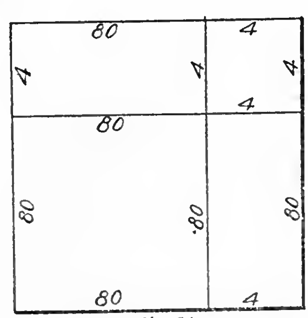
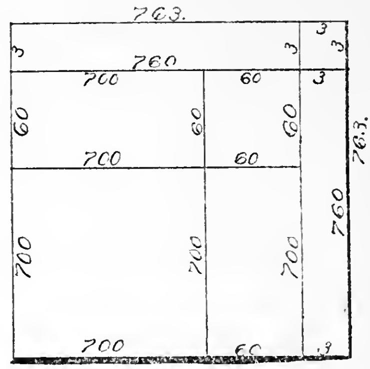


Fig 81.

2. Find the square of 763.

$763=700+60+3$

$$
\begin{aligned}
700^2 &= 490000\\
2\times700\times60 &= 84000\\
60^2 &= 3600\\
\hline
760^2 &= 577600\\
2\times760\times3 &= 4560\\
3^2 &= 9\\
\hline
763^2 &= 582169
\end{aligned}
$$



Note that if a number has one cipher on the right hand its square will have two, if it has two, its square will have four. In general the square of a number will have twice as many ciphers on its right as the number itself has.

Square Root.

The building up of squares suggests a method for extracting the square root of numbers.

The two right hand digits of a whole number are not used in finding the tens digit of its square root, four are not used in finding the hundreds, etc.

If a number is marked off into periods of two figures each beginning at the decimal point the figure to be used in finding the successive figures of its square root beginning at the left will be determined.

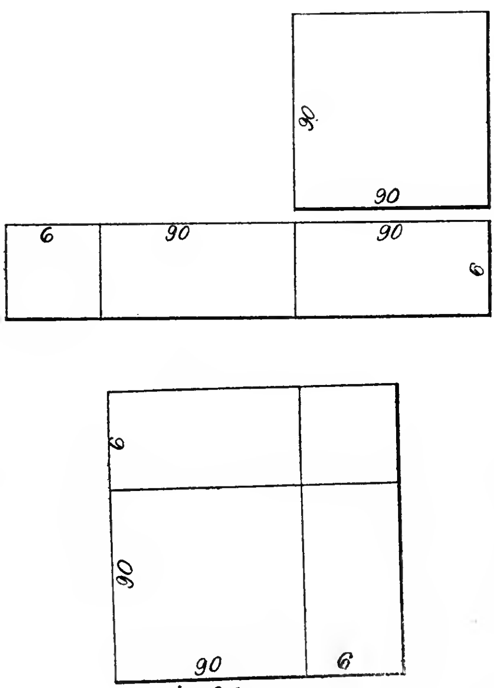


Fig. 84.

Extract the square root of 9216.

```
                  9 6
                -------
                 92'16
            90²=81 00
                -------
                |11 16
2 × 90 = 180    |
           6    |
        -----   |
         186    |11 16
                -------
```

Explanation.

The two right hand figures are cut off because the square of tens has two ciphers to the right. The square of the tens then in this case must not exceed 92. The tens figure is 9, for 9 squared is 81 the largest square not exceeding 92.

Subtracting the square of 9 tens, which is 8100, from 9216 leaves **1116.**

The 90 squared is represented by the upper diagram, and there is a remainder of 1116 square units to be added to it. In order to increase this square and still have a square, two rectangles each 90 long and a square must be added.

The two rectangles have a combined length of 180, and, since these comprise the greater part of what must be added, 180 may be used as a trial divisor. 1116 divided by 180 gives 6 as the trial figure and probable length of one side of the small square.

Placing the square along with the rectangles gives a total length of 186. Multiplying this length by 6 gives 1116, which is the number of square units that were remaining to be added. Hence the size of the completed square is 90 plus 6, or 96.

1. Find the square root of (1) 7225; (2) 5329; (3) 4489; (4) 1444; (5) 7396.

The work may be shortened by omitting the ciphers as shown below.

Find the square root of 8836.

$$
\begin{array}{r|l}
 & 9\ 4 \\
\hline
 & 88'36 \\
 & 81 \\
\hline
184 & \ \ 7\ 36 \\
 & \ \ 7\ 36 \\
\hline
\end{array}
$$

2. Find the square root of (1) 6084; (2) 5625; (3) 4761; (4) 6561; (5) 729, using the short method.

The same method is used when the answer contains more than two figures.

Extract the square root of 762129.

$$
\begin{array}{r|l}
 & 8\ \ 7\ \ 3 \\
\hline
 & 76'21'29' \\
 & 64 \\
\hline
167 & 12\ 21 \\
 & 11\ 69 \\
\hline
1743 & \ \ \ 52\ 29 \\
 & \ \ \ 52\ 29 \\
\hline
\end{array}
$$

3. Find the square root of (1) 245025; (2) 375769; (3) 249001; (4) 529984.

4. Square $\frac{5}{7}$; $\frac{14}{15}$; $\frac{9}{25}$; $\frac{15}{11}$; $3\frac{4}{5}$.

5. Find the square root of $\frac{4}{9}$; $\frac{81}{121}$; $\frac{121}{196}$; $\frac{49}{484}$; $\frac{25}{9}$; $4\frac{21}{25}$; $2\frac{2}{49}$; $26\frac{22}{49}$.

6. Square .3; .7; .02; .67; .009; .0468.

Compare the number of decimal places in the number with the number of decimal places in its square.

7. Find the square root of (1) .8649; (2) 51.84; (3) .0025; (4) 94.2841; (5) .000529; (6) .0000023409.

Sometimes the square root of a number can be found only approximately.

Find the square root of 38.7.

$$
\begin{array}{r|l}
 & 6.\ \ 2\ \ 2 \\
\hline
 & 38.'70'00 \\
 & 36 \\
\hline
122 & 2\ \ 70 \\
 & 2\ \ 44 \\
\hline
1242 & \ \ 26\ 00 \\
 & \ \ 24\ 84 \\
\hline
\end{array}
$$

In pointing off decimals begin at the decimal point.

8. Find to two decimal places the square root of (1) 24; (2) 6.4; (3) .76; (4) .144; (5) 9.04.

9. Find in rods one side of a

square field which contains (1) 10 A.; (2) 22½ A.; (3) 62½ A.

For additional examples in square root see the Grammar School Arithmetic, California State Series, pages 327-329.

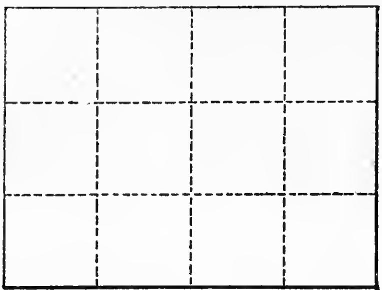


Fig. 85.

Draw a rectangle whose width is to its length as 3 is to 4. Divide it as shown in fig. 85. Into how many parts is the figure divided? What is the shape of each part? If the entire figure contains 588 sq. mm. what is the length of one side of one of the small divisions? Find the width and length of the figure.

10. Find the dimensions in rods of a field containing 4½ A., whose width is five-ninths of its length.

11. Find the perimeter in rods of a field containing 36 A., whose width is .9 of its length.

12. The entire surface of a cubical block is 2646 sq. cm. What is the length of one side?

13. The length, breadth, and thickness of a block are in the ratio of 4, 3, and 2, and its entire surface is 1300 sq. cm. Find its dimensions.

14. The length, breadth, and thickness of a solid are in the ratio of 5, 3, and 2, and its entire surface is 7502 sq. in. Find the sum of all its edges.

RIGHT TRIANGLES.

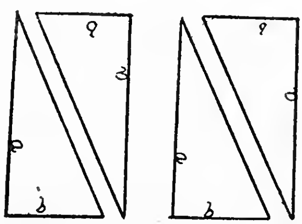


Fig. 76

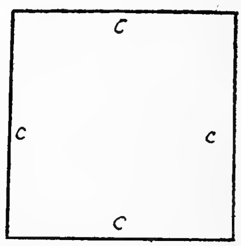


Fig. 77

1. Take the rectangle ab, and draw the diagonal c. This will divide the rectangle into two right triangles with base, a; perpendicular, b; and hypotenuse, c. Draw the square on c. Cut out and paste as shown in figures 76 and 77.

2. Cut out a square on a+b, a square on c, and four right triangles with sides, a, b, and c. Arrange and paste as shown in figure 78.

Are the outside lines of the right hand figure straight? How long are they? Compare the triangles with the rectangles.

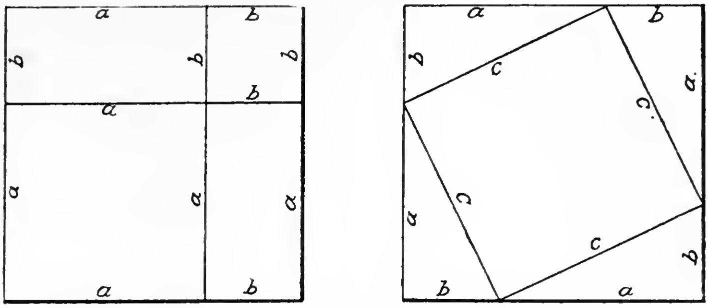


Fig. 78.

a^2+b^2+2 rect. $ab = c^2+2$ rect. ab.

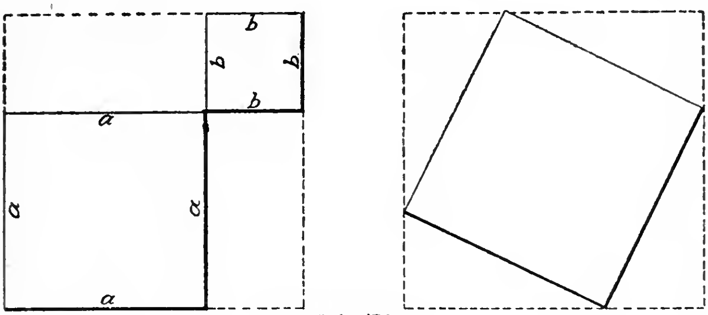


Fig. 79.

$a^2 + b^2 = c^2$

3. Remove the rectangles and triangles as shown in Fig. 79. Compare the remainders.

Arrange the figures as shown in fig. 80, and paste them in your book. The dotted lines show where the pieces were removed.

This proves that—**The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.**

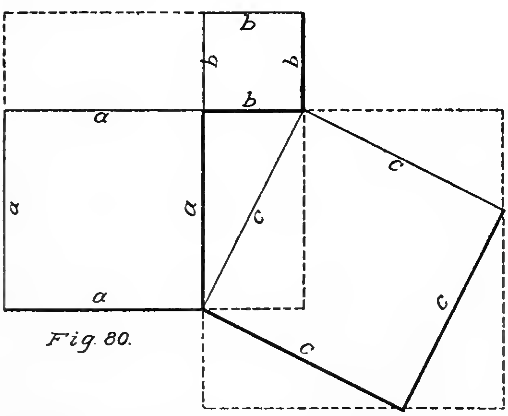


Fig. 80.

4. Find the distance diagonally across a room 20 by 30 ft.

5. Find the shortest distance from a lower to the opposite corner of a room 16 by 20 ft. by 14 ft. high.

A diagram will be very helpful in the solution of the problems which follow. For problems like that of spider and fly, use a pasteboard box. Cut along the corners and lay sides down on a level with the floor. The solution will then become evident.

6. A flag-pole 100 feet high stood 16 feet from a square-topped hen house 12 feet high and 14 feet wide. The pole fell, struck the upper edge of the hen house and broke. The lower part of the remaining portion tilted up without slipping, and the top struck the ground. How far from the base did the top of the pole strike? Ans., 94.89 ft.

7. A spider is on the floor three feet from one wall of the room and six feet from another. A fly is caught in a web in the nearest upper corner, fourteen feet above the floor. Find the shortest distance that the spider must travel along the floor and wall to reach the fly. Ans., 18.02 ft.

8. A room is 20 × 30 feet and 15 feet high. Find the shortest distance along the floor and walls from a lower corner to the opposite upper corner. Ans., 46.09 ft.

9. A log is 20 feet long and 16 inches in diameter. A string is wrapped about the log from end to end, each circuit being one foot from the others, the distance being measured lengthwise of the log. How long is the string? Ans., 86.13 ft.

10. Find the shortest distance along the surface of a block 10×12 ×15 inches from one corner to the one directly opposite. Ans., 26.62 in.

11. A hollow cylinder is 6 inches in diameter and 30 inches long. Find the shortest distance along the surface from a point on the upper edge to the opposite point on the lower edge. Ans., 31.4 in.

Parallelogram.

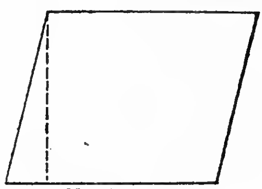


Fig. 44.

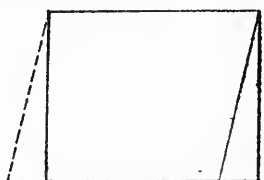


Fig. 45.

Chart III.

Draw a parallelogram 4 in. wide and 5 in. long as shown in figure 44. Cut it out and then using it as a pattern cut out another of the same size.

Cut off the corner of the second figure and place it on the right as shown in Fig. 45.

From this it is seen that—**A parallelogram is equal to a rectangle of the same base and height.**

1. Find the area of a parallelogram whose length is 12 in. and width 9 in.

2. A parallelogram is 8.5 ft. long and 224 in. wide. How many sq. ft. does it contain?

3. A parallelogram whose length is 24 ft. has an area of 336 sq. ft. How wide is it?

Triangle.

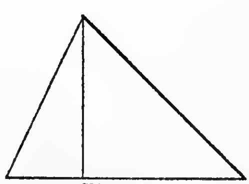


Fig. 46.

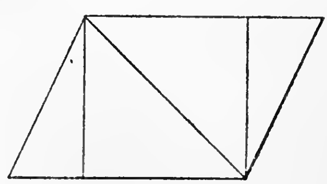


Fig. 47.

Chart IV.

Draw a triangle with base 5 in. and height 4 in. as shown in Fig. 46. Cut it out and cut out two more of the same size. Place the second and third as shown in Fig. 47.

Note that Fig. 47 is a parallelogram having the same base and altitude as the triangle in Fig. 46, but twice the area.

From this it is learned that—**A triangle equals half a parallelogram of the same base and height.**

Make a rule for finding the area of a triangle.

1. Find the area of a triangle whose base is 52 in. and height 28 in.

2. A triangle has a base of 8.74 ft. and height of 1.26 ft. How many sq. ft. in it?

3. Find the height of a triangle whose base is 3.5 yd. and area 14 sq. yd.

4. A triangle whose area is 275 sq. yd., is 25 yd. high. What is its base?

5. Find the number of sq. cm. in the surface of Fig. 48.

6. Make necessary measurements in Fig. 49 and find the number of sq. mm. it contains.

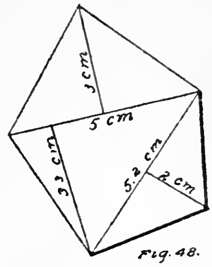


Fig. 48.

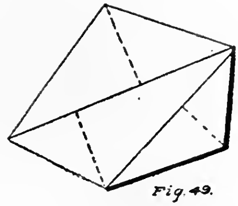


Fig. 49.

In an isosceles or an equilateral triangle, the perpendicular from the vertex to the base strikes the base at its midpoint.

7. Find the altitude and area of an isosceles whose equal sides are each 13 feet and base 10 feet. Ans., Area 60 sq. ft.

8. The perimeter of an isosceles triangle is 100 rd. and its base is 40 rd. Find the area. Ans., 447.2 sq. rd.

9. Find the altitude and area of an equilateral triangle each of whose sides is 30 feet. Ans., Area 389.7 sq. ft.

10. The area of an isosceles triangle is 4800 sq. ft. and its base is 120 ft. Find the length of one of the equal sides. Ans., 100 ft.

If a line is drawn from each vertex of a regular hexagon (six sided figure) to the center, the figure will be divided into six equilateral triangles.

11. Find the area of a regular hexagon each of whose sides is 50 ft. long. Ans., 6495 sq. ft.

Trapezoid.

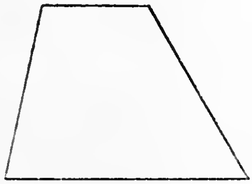


Fig. 50.

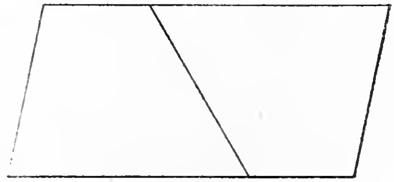


Fig. 51.

Chart V.

A trapezoid has only two of its sides parallel. These are called bases.

Draw a trapezoid whose bases are 5 in. and 4 in. and height 3 in. as shown in Fig. 50. Cut it out and cut out two more of the same size. Place the second and third figures as shown in Fig. 51.

Note (1) that Fig. 51 is a parallelogram, (2) that it has the same altitude as the trapezoid in Fig. 50, (3) that the base of the parellogram equals the sum of the bases of the trapezoid, and (4) that the area of the parallelogram equals twice the area of the trapezoid.

This shows that — **A Trapezoid equals half a parallelogram of the same height and having a base equal to the sum of the base of the trapezoid.**

To find the area of a trapezoid, multiply the sum of its bases by its altitude and divide the product by two.

1. Find the number of sq. cm. in a trapezoid whose bases are 12 cm. and 10 cm. and height 9 cm.

2. The bases of a trapezoid are 25 yds. and 15 yds. and the height 562 yds. How many sq. yd. does it contain?

3. The bases of a trapezoid are 15 yd. and 9 yd. and its area 167 sq. yd. What is its height?

4. A trapezoid contains 324 sq. cm. Its height is 12 cm. and upper base 24 cm. Find the lower base.

5. Make the necessary measurements and find the number of sq. mm. in Fig. 52.

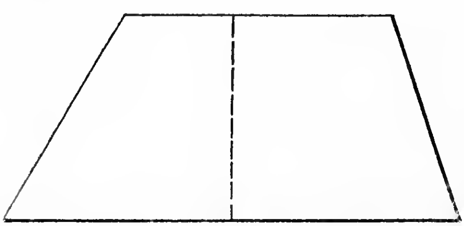


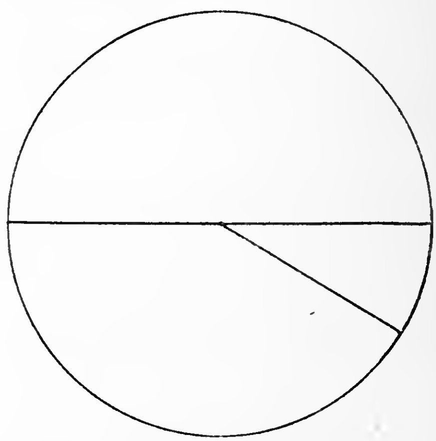
Fig 52

6. The roof of a certain house consists of two equal trapezoids whose lower and upper bases are 48′ and 32′ respectively and altitudes 25′, and two triangles whose bases are each 16′ and altitudes 25′. Find the cost at $2.75 M. of shingles to cover it. The shingles are to be laid 4″ to the weather.

7. The roof of an L shaped house consists of two triangles, two parallelograms, and two trapezoids each with an altitude of 18 feet. The bases of the triangles are 30′, of the parellograms 14′, and of the trapezoids 44′ and 14′ respectively. How many bunches of shingles laid 4½″ to the weather are required to cover it?

Circle.

Open your dividers, place one point on the paper, and revolve the other about it making a mark. The figure formed is a circle, the bounding line is the circumference, the fixed point on which one point of the dividers was placed is the center, a straight line drawn from the center to the circumference is a radius, a line drawn thru the center and terminated by the circumference is a diameter.



The following rules are proved in geometry.

Circumference of a circle equals 2×3.1416 times the radius.

Area of a circle equals 3.1416 times the square of the radius.

The Greek letter π (pi) is used in mathematics to stand for 3.1416 ($3\frac{1}{7}$ nearly). More briefly these rules are written

$$\text{Circum } \bigcirc = 2\pi r$$
$$\text{Area } \bigcirc = \pi r^2$$

1. Find the circumference of a circle whose diameter is 15 feet.

2. A grindstone is 2 feet in diameter. Find its circumference in inches.

3. A tree is 30 feet in circumference. Find its diameter.

4. Find the diameter in feet of a circular race track a mile in length.

5. Find the number of square inches of tin in a 6 in. joint of stovepipe 20 inches long, making no allowance for seam.

6. Find the area of a circular grass plot 16 feet in diameter.

7. A horse is staked with a rope 40 feet long. Over how much ground can it graze?

8. How much lawn can be watered with a 50 foot hose attached to a hydrant, if the water is thrown 12 feet beyond the end of the hose?

9. A galvanized tank is 8 feet in diameter and 7 feet high. How many square feet of iron are used in making it?

10. Four equal circles 10 inches in diameter are cut from a piece of pasteboard 20 inches square. How much of the pasteboard remains? Find the area of each piece.

11. A tree is three feet in diameter. Find the side of a square that will have the same area.

12. A horse is tied to the corner of a barn 30 × 40 with a rope 70 feet long. Over how much ground can the horse graze? Ans., 13508.88 sq. ft.

13. Over how much ground can the horse graze if tied to the middle of the longest side of the same barn with the same rope?
Ans., 12252.24 sq. ft.

14. Over how much ground can the horse graze if tied to the middle of the shortest side of the same barn with the same rope? Ans., 12802.02 sq. ft.

15. A rope is stretched between two stakes 25 feet apart. A horse is tied with a rope 30 feet long fastened to a ring which slips back and forth on the first rope. Over how much ground can the horse graze? Ans., 4327.44 sq. ft.

16. A horse is tied with a rope 50 feet long fastened to the top of a pole 12 feet high. Over how much ground can the horse graze?

SOLIDS AND VOLUMES.

A cube is a soild whose faces are equal squares. (Fig. 53.) How many faces has a cube? How many edges? How many verticles (corners)?

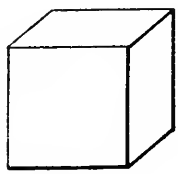


Fig 53

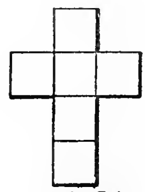


Fig. 54.
½ size.

If each edge of a cube is one foot, the figure is a cubic foot (cu. ft.). If each edge is one inch, it is a cubic inch (cu. in.), etc.

1. Draw the surfaces of a cu. in. as shown in Fig. 54. How many sq. in. on the surface?

2. How many sq. in. on the surface of a cube whose edge is 3 in.?

3. If the area of a cube is 150 sq. in., how long is the edge?

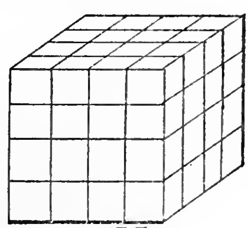


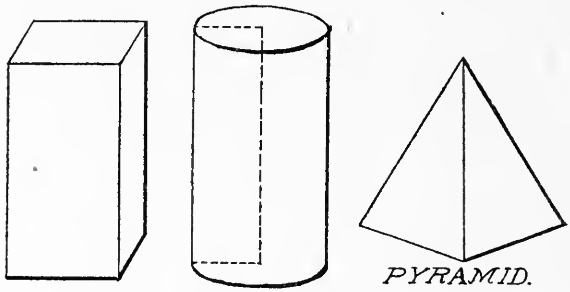
Fig. 55.

4. Draw a cube whose edge is 4 inches. Mark it as shown in Fig. 55. What is the size of each of the small solids marked off? How many cu. in. in one row of the top layer?

How many rows in that layer? How many layers in the whole cube? How many cu. in. in the cube?

5. How many cu. in. in a cube whose edge is 6 in.?

6. How many cu. in. in a cu. ft.?



PRISM. CYLINDER. PYRAMID.

7. How many cu. yd. of earth is removed in digging a cellar 3 yd. each way?

8. How many cu. ft. in a cu. yd.?

A rectangular solid is a solid with six rectangular faces.

Draw a rectangular solid whose length, breadth and thickness are 4 in.; 3 in. and 2 in., respectively. Draw lines dividing it. Find the sum of its edges. Draw its faces as in exercise 1. Find its area. How many cu. in. does the solid contain?

When we find the number of cubes that a solid contains we say we find its volume.

9. Find the volume in cu. in. of a solid 4 in. by 5 in. by 7 in. Find the number of sq. in. in its surface.

10. Find the volume in cu. ft. of a solid 25 ft. by 3 ft. by 15 yd.

PRISMS AND CYLINDERS.

A **right prism** is a solid whose bases are polygons and whose sides are rectangles.

Mention some objects that have the form of a prism. Construct of cardboard a square prism whose base is 4 inches on each side, and whose height is 5 inches. Paste the sides and bottom and cut the top off. How many sq. in. of cardboard are required? How many cu. in. will make one layer on the bottom? How many cu. in. will the prism hold?

If a rectangle is made to revolve about one of its sides, two of its sides describe circles, and the other describes a curved surface. The solid thus formed is a cylinder.

Construct of cardboard a cylinder, making the base 4 inches in diameter and the height 5 inches, omit the top.

To find the volume of any prism —Multiply the base by the height.

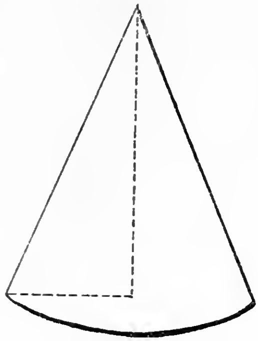
1. The sides of a right prism are three feet by four feet, and its height is twelve feet. Find its volume and its lateral area.

2. A hollow prism is three by five feet on the outside and two by three feet on the inside, and it is forty feet long. Find its volume and the area of its outer and inner lateral surfaces.

3. How much sheet iron is used in making a six inch joint of stove pipe thirty inches long, allowing four-tenths of an inch for the seam?

4. How many sq. cm. of tin are required to make a lard can 12 cm. in diameter and 14 cm. high, with a lid whose rim slips down 1 cm.? Draw the developed surface. How many cu. cm. of lard will the can hold? What will the lard weigh if 1 cu. in. of lard weighs .84 g.?

5. Find the weight of a piece of copper wire 4 mm. in diameter and 75 m. long. (1 cu. cm. of copper weighs 8.6 g.).



CONE.

6. Find the weight of a circular piece of gold 5 cm. in diameter and 6 mm. thick. (1 cu. cm. of gold weighs 19.3 g.)

Pyramids and Cones.

Pyramid—A solid whose base is a polygon and whose sides are triangles meeting in a common point is a **pyramid.**

Construct of cardboard a square pyramid with the sides of the base each 4 inches and perpendicular its height 5 in. Paste the sides of the pyramid together and cut off the bottom.

Fill the pyramid with sand and then pour the same into the prism which you made. Fill the pyramid again and empty it. It will take three pyramidfuls to fill the prism.

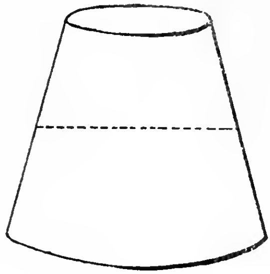
To find the volume of a pyramid—Multiply the base by one-third of the height.

7. How many cu. in. will the pyramid which you have made hold?

8. Find the volume of a square pyramidal block of wood with base 10 in. square and height 12 in.

Cone—If a right triangle is made to revolve about one arm, the other arm describes a circle, and the hypotnuse describes a curved surface. The solid generated is a **cone.**

Construct of cardboard a cone whose base is 4 in. in diameter, and height 5 in. Paste the sides of your cone and omit the bottom.



FRUSTUM

Fill the cone with sand and empty into the cylinder of the same dimensions until the cylinder is full. Three conefuls are required.

To find the volume of a cone—Multiply the base by the height and divide by three.

9. How many cu. in. in the cone which you have constructed?

10. A conical block of wood is 8 in. in diameter at the base and 15 in. long. How many cu. in. of wood does it contain?

Frustums.

The part of a pyramid or cone contained between the base and a plane parallel to the base is a **Frustum.** The mid-base is the base midway between the upper and lower bases.

To find the volume of a frustum—Add together the upper base, the lower base, and four times the mid-base, multiply the sum by the height and divide the product by six.

Note—This rule is exact, and also gives correctly the contents of an enbankment, a ditch, or a frustum of a wedge.

To find the lateral surface of a frustum—Multiply the perimeter of the mid-base by the slant height.

Any line of the mid-base is a mean between the corresponding lines of the upper and lower bases.

Find the volume of a frustum with rectangular bases, whose lower base is 18×24 inches, upper base 10×16 inches, and height 30 inches.

Solution:

Length of mid-base (24 in.+16 in.) $\div 2 = 20$ in.

Width of mid-base (18 in.+10 in.) $\div 2 = 14$ in.

Lower Base=	$24 \times 18 =$	432
Upper Base=	$16 \times 10 =$	160
$4 \times$ mid-base=	$4 \times 20 \times 14 =$	1120
Sum		1712

$1712 \times 30 \div 6 = 8560$

Hence the volume is 8560 cu. in.

1. A dishpan is 36 cm. in diameter at the top, 20 cm. at the bottom, and 12 cm. deep. How many liters of water are required to fill it? Ans., 7.59 l.

2. A telegraph pole is 14 in. square at the base, 8 in. square at the top and 48 ft. long. How many cubic feet does it contain? Ans., 41.3+ cu. ft.

3. The base of a chimney 90 ft. high is 16 ft. square on the outside and 10 ft. square on the inside. At the top the chimney is 8 ft. square on the outside and 6 ft. square on the inside. How many cubic feet of masonry in the chimney? Ans., 7560 cu. ft.

4. A round hollow iron pillar, 18 ft. long, is 16 inches in diameter at the base, and 12 at the top. The iron is three inches thick at the base and two at the top. How much will the pillar weigh if a cubic inch of iron weighs .26 lb. Ans., 5116.5+ lb.

A wedge may be considered a frustum in which the upper base is a line and its area O. The short line of the mid-base is half the corresponding line of the lower base. A hip roof together with the upper ceiling forms a kind of wedge, the ridge being the upper base.

5. The base of a wedge 9 in. long is 2 in. by 3 in., the edge is 2 in. long and parallel to the long side of the base. Find its volume. Ans., 24 cu. in.

6. Find the number of cubic feet enclosed by the hip roof of a house 36 ft. by 44 ft., one-third pitch. Ans., 6912 cu. ft.

One-third pitch means that the heights of the ridgepole above the upper wall plate is one-third of the width of the building. The ridgepole of a hip roofed house equals the difference between the length and width of the building. In the building described in the last problem the height of the roof is 12 ft. and the ridgepole is 8 feet long.

7. Find the volume of a regular hexagonal pyramid whose altitude is 30 feet and each side of whose base is 40 feet. Ans., 41568 cu. ft.

8. Each side of the lower base of a frustum of a pyramid is 8 feet, each side of the upper base is 6 ft., and the altitude is 12 ft. Find the volume. Ans., 592 cu. ft.

9. A tank in the form of a frustum of a cone and constructed of two-inch lumber has the following outside measurements: Height 5 ft. 8 in., diameter of base 6 ft., diameter of top 6 ft. 8 in. How many gallons will it hold? Ans., 1161.5 gal.

Levees, canals, railway cuts and embankments may be considered as frustums lying on one side. Their volumes are found by calculating in sections of 50 ft. or 100 ft. in length. The end sections of a levee are usually trapezoids.

10. Find the volume of the fifty foot section of a levee with the dimensions here given.

width: top	width: bottom	ht.
10′	18′	6″
10′	22′	9″

Ans., 5650 cu. ft.

Sphere.

Surface of a sphere $= 4\pi r^2$

Volume of a sphere $= \frac{4}{3}\pi r^3$ or $\frac{1}{6}\pi d^3$

Note that a sphere has just four times the area of a circle of the same radius.

11. Find the area of a sphere whose diameter is 1 foot.

12. Find the radius of a sphere whose area is 1 sq. m.

13. The diameter of the earth is 7918 miles. Find its circumference and its area.

Area, 196961744.4 sq. mi.

Circumference, 24875.1888 mi.

RATIO.

Ratio is the indicated quotient arising from dividing the first of two given numbers by the second. The ratio of 6 to 9 is $6 \div 9 = \frac{6}{9}$, or $\frac{2}{3}$, the ratio of 12 feet to 7 feet is $12 \div 7$ or $\frac{12}{7}$. A ratio may be indicated by the use of a colon, as 5:9, or by a fraction, as $\frac{5}{9}$. In either case it is read the ratio of 5 to 9. The fraction form is preferable.

The first term of a ratio is called the antecedent and the second the consequent.

Division, fraction, and ratio mean practically the same thing. Thus $11 \div 8$, $\frac{11}{8}$, and the ratio $\frac{11}{8}$ or 11:8 are only three ways, getting the same results, $1\frac{3}{8}$. Note that **dividend, numerator,** and the antecedent are corresponding terms, as are divisor, denominator, and consequent. The corresponding values are called quotient, fraction, and ratio respectively.

The following principles are easily verified:

1. Multiplying the dividend multiplies the quotient by the same number.

2. Dividing the dividend divides the quotient by the same number.

3. Multiplying the divisor divides the quotient by the same number.

4. Dividing the divisor multiplies the quotient by the same number.

5. Multiplying or dividing both dividend and divisor by the same number does not change the value of the quotient.

Restate these principles substituting numerator for dividend, denominator for divisor, and fraction for quotient.

Restate them substituting the words antecedent, consequent, and ratio for dividend, divisor and quotient respectively.

Note (1) that ratio can exist only between like quantities, (2) that the value of a ratio is an abstract number.

Feet and yards or gallons and quarts can be compared only by changing to the same denomination. Hogs and sheep can be compared by weighing or valuing. Fractions can best be compared by reducing to a common denominator and comparing their numerators.

Find the ratio of:

1. 25 to 45.

2. 78 to 18.

3. $4\frac{2}{3}$ to $5\frac{1}{2}$.

4. 65 lb. to 39 lb.

5. 7 ft. to 28 in.

6. 19 pt. to 4 gal.

7. 2 yd. to 7 yd. 1 ft.

8. 4 cords of wood @ \$7 to 20 sheep @ \$$3\frac{1}{2}$.

9. 6748 lb. prunes @ $1\frac{1}{2}$c to 4685 lb. peaches @ $1\frac{3}{4}$c.

10. 465 hogs averaging 225 lb. @ 7c lb. to 160 steers averaging 525 lb. @ 11c lb.

11. $2\frac{3}{7}$ miles to 1 mile $167\frac{2}{3}$ rd.

Proportional Parts.

Divide \$17500 between Charles and Helen so that their shares shall have the ratio of 3 to 4.

The conditions of the problem will be satisfied if the money is divided into seven equal parts and Charles is given three and Helen four parts.

Solution.

Let 3 parts equal Charles' share.

And 4 parts equal Helen's share.

Then 7 parts equal the whole amt.

7 parts equal \$17500.

1 part equals \$2500.

3 parts equal \$7500, Charles' share.

4 parts equal \$10000, Helen's share.

The value of one of the equal parts may be represented by some letter as x.

The ages of Reuben, Silas and Joseph have the ratio 3, 4, and 5, and the sum of their ages is 36 years. What is the age of each?

Let 3x equal Reuben's age.

Then 4x equals Silas's age

and 5x equals Joseph's age.

12x equals sum of their ages.

12x equals 36 years.

x equals 3 years.

3x equals 9 years, Reuben's age.

4x equals 12 years, Silas's age.

5x equals 15 years, Joseph's age.

1. Gunpowder is made of 2 parts sulphur, 1 part of saltpetre, and 3 parts charcoal. How many pounds of each of other ingredients must be used with 250 pounds of saltpetre?

2. How many pounds each of sulphur, saltpetre, and charcoal will be required to make 1000 pounds of powder?

3. The length of a field is to its width as 7 to 5 and its perimeter is 120 rods. Find its dimensions and area.

4. The length of a field containing 630 acres is to its width as 7 to 4. Find the length and width of the field. Draw a diagram.

5. The length, breadth and thickness of a rectangular solid have the ratio of 5, 4, and 3, and the sum of its edges is 912 inches. Find the dimensions of the solid.

6. The length, breadth and thickness of a rectangular solid are in the ratio of 5, 3, and 2, and its surface contains 1550 square inches. Find its dimensions.

7. The length, breadth and thickness of a rectangular solid have the ratio of 6, 5, and 2, and its volume is 20580 cubic inches. Find its area. Illustrate.

8. John is 12 years older than Henry. In seven years Henry will be three-fourths as old as John. What is the age of each now?

9. William is 18 years old and Henry is 10. In how many years will Henry's age equal five-sevenths of William's? Ans., 10 yr.

10. Mr. Grant is 52 years old and his wife is 40. When they were married Mrs. Grant's age was $\frac{5}{8}$ of Mr. Grant's. How long have they been married? Ans., 20 yr.

$\frac{3}{4}$ of the time past midnight = $\frac{6}{7}$ of the time to noon. What is the time?

Solution.

$\frac{3}{4}$ of the time past midnight = $\frac{6}{7}$ of time to noon.

$\frac{1}{4}$ of time past midnight = $\frac{2}{7}$ of time to noon.

$\frac{4}{4}$ of time past midnight = $\frac{8}{7}$ of time to noon.

Ratio 8 to 7.

Let 8x=time past midnight

Then 7x=time to noon

15x=the sum

15x=12 hours

x=$\frac{4}{5}$ of an hour

8x=$6\frac{2}{5}$ hours or 6 hours and 24 min. Ans., 6:24 a. m.

11. A pole 76 ft. long was broken in such a manner that $\frac{3}{5}$ of the length of one piece equals $\frac{2}{3}$ of the length of the other. How long is each piece? Ans., 40 ft.; 36 ft.

12. Two-thirds of Mary's age equals $\frac{3}{5}$ of Homer's and the difference in their ages is 2 years. What is the age of each?

In the first eight problems below it is assumed that A, B and C own equal shares of the sheep.

It is best to find the total expense and then find each partner's share of it. For example, in the first problem, since C's share of the expense is $50, the total expense is $150, and the pasturage is therefore worth $1.50 an acre. In partnership each partner must pay all his own private expenses.

13. A, B and C have a flock of sheep of which each owns a third. A has 60 acres, and B 40 acres of pasture land. The sheep eat the pasturage and C pays $50. Find A's and B's share of the money. Ans., A, $40; B, $10.

14. A has 80 acres, B 40 acres, and C 30 acres of pasture land on which the sheep graze. B and C agree to pay A $15. How much should each pay? Ans., B, $5; C, $10.

15. A has 60 acres, B 25 acres, and C 15 acres of pasture land on which the sheep graze. C pays $13.75. How much should A and B pay or receive? Ans., A rec. $20; B pays $6.25.

16. A has 125 acres and B 60 acres of pasture land and they rent from other parties 115 acres. The sheep eat the pasturage which is valued at 75 cts. an acre. How much should A, B and C each pay or receive? Ans., A would receive $18.-75; B pay $30; C pay $75.

17. A has 75 acres and B 25 acres of land on which the sheep are pastured. C is allowed $15 for looking after the sheep. If the pasturage is reckoned at 60 cts. an acre, how much should each pay or receive? Ans., A rec. $20; B and C each pay $10.

18. A has 75 acres and B 50 acres of pasture on which the pasturage is worth 30 cents an acre. A flock of sheep, of which A, B and C own equal shares, eat the pasture. C is allowed $20 for caring for the sheep. How should the bills be settled?

19. A has 80 acres of land with pasturage worth 75 cents an acre. B has 60 acres of land with pasturage worth $1.25 an acre. A, B and C have the same number of sheep to graze on the land. C is allowed $15 for his work. How shall they settle the bills? Ans., A rec. $10; B rec. $25; C pays $35.

20. A has 40 acres of land worth 50 cts. an acre, B has 65 acres worth $1.00 and C 20 acres worth $1.25 an acre. A and C take care of the sheep @ $5 each. How should they settle the bills? Ans., A pays $15; B rec. $25; C pays $10.

21. A owns 500 sheep, B 700, and C 800. A has 75 acres, B 100 acres, and C 140 acres of land, the pasturage of which is valued at 60 cents an acre. C takes care of the sheep for $15. If the sheep eat the pasturage off all the land, how should the bills be settled? Ans., A pays $6; B pays $11.40; C rec. $17.40.

22. A, B and C are partners in a store in the rates of 4, 3 and 5. At the end of a year they find that the gross gain is $5000 and the store expenses $1400. Each has spent $900 for private expenses during the year. Find the conditoin of each man's account. Ans., A gains $300; C gains $600; B gains $0.

23. A, B and C are partners in the ratio of 5, 4 and 3. Their gross gains are $5700. At the end of a year their store expenses including salary are $2100. A's salary is $1000 a year, and their private expenses are $850 each. What is the gain of each at the end of a year? Ans., A gains $1650; B gains $350; C gains $50.

24. A and B own a store in the ratio of 2 to 5, and each is allowed a salary of $800 a year for his services. The yearly gross gain is $6500, the store expenses not including the salaries are $840, and each spends $2500 a year for private expenses. What will be the gain or loss of each in 5 years?

Ans., B gains $6000; A loses $2700.

25. A and B own a store in the ratio of 5 to 2. A is allowed $1500 and B $1000 for personal services. The gross gain is $6500. The store expenses, including salaries, is $3,-350. A's private expenses are $2500 and B's are $2000. Find the gain or loss of each. Ans., B loses $100; A gains $1250.

26. A machine is sold for $100, one-fifth of the sales being profit. In manufacturing the machine the cost of labor is to the cost of the material as 7 is to 9. If the cost of labor advances one-fifth and that of material falls one-fifth, and the machine sells for $100 what will the profit be? Ans., $22 gain.

27. Mesdames Brown, Jones and Smith prepared a dinner. Mrs. Brown furnished fruit and vegetables valued at $3, Mrs. Jones furnished meat and pastries valued at $4.50. Bread was purchased and dishes rented at an expense of $2.50. Mrs. Smith did the work for $5. The Brown family numbered 9; the Jones family 10, and the Smith family 11 How should the bills be settled? Ans., Mrs. J. pay $.50; Mrs. S. pay $.50; Mrs. B. pay $1.50.

28. Fast, Slow and Steady owned a dairy in the ratio of 7, 6, and 9. Fast received $30 a month wages; Slow, $18; and Steady, $25. The feed and other expenses cost $120. The receipts for the month were $270. Find each man's share. Ans., Fast, $54.50; Slow, $39.00; Steady, $56.50.

29. A, B and C are partners in farming. A furnishes 250 acres of land which rents at $3 an acre, B furnishes seed and provisions worth $500, and C furnishes stock, hay and implements, the use of which is valued at $450. Each is allowed $200 a year for wages. The crop sells for $3140, and the personal expenses are $450, $350, and $300 respectively. Find the gain or loss of each, the profits being shared equally. How much money will each have at the end of the year? Ans., A has $780; B, $630; C, $630.

Proportion.

Simple proportion consists of two pairs of equal ratios. Thus, $\frac{5}{9}$ equal $\frac{15}{27}$, the ratio 27 yards to 18 yards equals the ratio of $9 to $6 for $\frac{27}{18}$ equals $\frac{9}{6}$.

A proportion may be expressed as follows: $\frac{16}{10} = \frac{24}{15}$, $16:10 = 24:15$, or $16:10 :: 24:15$. The first form given is preferable.

The first and fourth terms of a proportion (16 and 15 in the above) are called the extremes and the second and third (10 and 24) the means.

In every proportion the product of the extremes equals the product of the means.

One quantity varies as another when an increase or decrease in one causes a corresponding increase or decrease in the other, that is when if one is multiplied or divided by a number the other is multiplied or divided by the same number. The distance one can travel varies as the time when the rate is the same. The amount of grain produced will vary as the amount of land if the production per acre is the same. It will vary as the amount produced on one acre on the same number of acres.

One quantity varies inversely as another when if one is multiplied or divided by any number the other is divided or multiplied by the same number. The time required to travel a given distance varies inversely as the rate. The number of articles that can be bought with a given amount of money varies inversely as the price of one article. The number of men required to do a piece of work varies directly as the amount of work and inversely as the time in which it must be done.

If a train travels 486 miles in 14 hours, how many hours will it take to travel 3000 miles at the same rate?

Solution.

hours.	miles.
14	486
?	3000

Let x equals time required to travel 3000 mi.

Then $x/_{14} = {}^{3000}/_{486}$

$$x = \frac{14 \times 3000}{486}$$

86.4 plus hr. Ans.

If 75 men can do a piece of work in 30 days, how many men would be required to do it in 18 days?

men.	days.
75	30
?	18

Let x equal number of men to do the work in 18 da.

$x/_{75} = {}^{30}/_{18}$

$x = 75 \times {}^{30}/_{18}$ equals 125.

125 men Ans.

The work may be shortened by omitting the first equation.

If 3800 sacks of wheat are produced on 320 acres of land, how many acres of land of the same grade are required to produce 12000 sacks?

sacks.	acres.
3800	320
12000	?

Let x equal acres required to produce 12000 sacks.

$$\text{Then } x = \frac{320 \text{ A.} \times 12000}{3800}$$

$x = 1010.52$ A.

These solutions suggest the rule.

RULE: Place the unknown term equal to the like known multiplied by the ratio of the other two, that ratio being made greater than one when the answer should be greater, and less than one when it should be less than the term like the required answer.

1. If 17 acres of prunes bring a net income of $3500, what should be the net income of a similar crop on 75 acres.

2. If land which nets its owner $9 an acre yearly is valued at $75 an acre, what is the value of land which nets $200 an acre yearly?

3. If a train makes a certain distance in 27 hours traveling 35 miles an hour, how long will it take an aeroplane traveling 76 mi. an hour to make the same distance?

4. Twenty-five tons of ore of a certain mine produce metal worth $417.65, how much of the same ore is required to yield metal worth $4000?

5. A contractor engaged to complete a piece of road work in 30 days. He employed 26 men, and found at the close of 18 days that ½ the work was done. How many additional men must be employed that the work may be completed in the required time?

6. A ship started on a cruise of 90 days with provisions for its 125 men. At the end of 70 days 25 men left the ship. How long will the provisions last the remainder?

Compound Proportion.

Sometimes one ratio depends upon the combined effect of two or more ratios. The distance traveled depends on both the rate and the time. The value of a crop depends upon both the price and the amount produced. Ratios thus combined constitute a compound ratio and the resulting proportion a **compound proportion.**

The solution of a compound proportion does not differ materialy from that of a simple proportion.

Write the unknown term equal to the like known multiplied by the ratios of the other like terms. The terms of each ratio must be arranged as if the result depended on it alone.

If it costs $85 to floor a room 14′ by 24′, how much will it cost to

floor a room 16′ by 22′ with flooring of the same price?

cost.	width.	length.
\$85	14	24
?	16	22

Let x = cost of flooring room 16′ by 22′.

Then $x = \$85 \times {}^{16}/_{14} \times {}^{22}/_{24}$

$x = \$89.10$ Ans.

If 200 men construct 18 miles of road in 55 days, how long will it take 220 men to construct 27 miles?

men.	days.	miles.
200	55	18
220	?	27

Let x = days it takes 220 men to construct 27 miles.

Then $x = 55 \text{ days} \times {}^{200}/_{220} \times {}^{27}/_{18}$

$x = 75$ days Ans.

220 men can construct the road in less time than it takes 200 men, hence the ratio, $^{200}/_{220}$, must be less than 1.

It will take longer to construct 27 miles than 18 miles, hence the ratio, $^{27}/_{18}$, must be greater than 1.

1. If the peach crop from 25 acres brings \$3400 when the dried fruit is sold at 7c a pound, what is the value of a similar crop on 45 acres sold at 6c a pound?

2. If 4200 tons of iron ore produce \$3430 when iron is \$36.80 a ton, what is the value of 3500 tons of like ore when iron is \$42.50 a ton?

3. If a reservoir 600 feet long, 350 feet wide and 9 feet deep holds 450000 barrels, how much will a reservoir 800 feet long, 375 feet wide and 7 feet deep hold?

Similar Figures.

Similar surfaces and solids are those which have the same shape. Their corresponding angles must be equal and their corresponding lines are proportional.

It is true of similar figures that (1) their corresponding lines are proportional, (2) the areas of similar are to each other as the squares of their corresponding lines, (3) the volumes of similar figures are to each other as the cubes of their corresponding lines.

The dimensions of a block of marble are 8 in., 6 in., and 4 in., the cost of polishing it \$.65, and its weight is 18.26 lb. The length of a similar block is 42 in. Find the width, thickness, and weight of the second block and the cost of polishing its surface.

l.	w.	th.	cost	wt.
8″	6″	4″	\$.65	18.26
42	?	?	?	?

w. equals $6'' \times \frac{42}{8}$,

th. equals $4'' \times \frac{42}{8}$,

cost equals $\$.65 \times \frac{42^2}{8^2}$,

wt. equals $18.26 \text{ lb.} \times \frac{42^3}{8^3}$.

1. A cone is 6 inches in diameter and 7 inches in altitude. What is the height of a similar cone 11 inches in diameter?

2. If it takes 24 thousand shingles to cover a roof 40 by 65 feet, how many will it take to cover both sides of a similar roof 30 feet wide?

3. If a man 6 feet tall weighs 180 lb. what did Goliath who was 10.5 feet tall weigh if he was of similar proportions?

4. If a 6 inch cannon ball weighs 39.5 pounds what is the weight of a 14 inch ball of the same material?

Longitude and Time.

The earth revolves on its axis from west to east, causing the succession of day and night. It makes a complete revolution in 24 hours; hence, at a given place it is 24 hours from sunrise to sunrise, or from noon to noon.

Show that the following statements are true:

The earth turns 360° in 24 hr.
" " " 15° " 1 hr.
" " " 1° " 4 min.
" " " 15′ " 1 min.
" " " 1′ " 4 sec.
" " " 15″ " 1 "

Stand with your face to the south and point toward the place where the sun is seen at noon. Will it be forenoon or afternoon with persons living east of you? West of you? It is noon now at some place. Is that place east or west of here? Where was it noon an hour ago? Where will it be noon an hour hence?

The time of day is later east and earlier west of a given place.

Examine the following statements and tell why they are true:—

A difference of 15° in lon. makes a difference of 1 hour in time.

A difference of 15′ in lon. makes a difference of 1 min. in time.

A difference of 15″ in lon. makes a difference of 1 sec. in time.

Difference in longitude may be changed to difference in time by dividing by 15 and calling the quotient hrs., min. or sec., according as the dividend is degrees, minutes, or seconds.

Difference in time may be changed to difference in longitude by multiplying by 15 and calling the result degrees, minutes, or seconds according as the multiplicand is hours, minutes, or seconds.

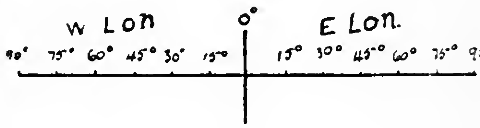
Thru the relation of longitude and time a captain determines the longitude of his ship at sea, and the longitude of a place on the land is found.

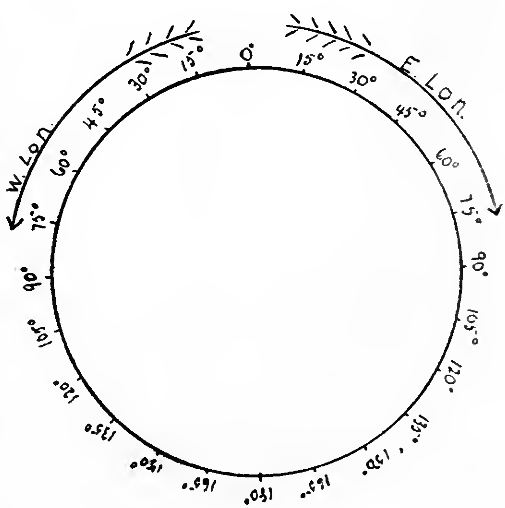
1. Find the difference in time between (1) 3:30 a. m. and 7:15 a. m.; (2) 6:25 a. m. and 4:45 p. m.; (3) 1:10 p. m. and 11:55 p. m.; (4) 2:55 a. m. and 9:12 p. m.; (5) 7:18 p. m. and 4:27 a. m. next day; (6) 8:20 a. m. and 10:40 p. m. next day; (7) 11:27 a. m. and 6:45 a. m. next day; (8) 6:15 a. m. and 4:35 p. m. next day; (9) 3:30 p. m. and 8:15 a. m. preceding day; (10) 9:27 p. m. and 5:15 a. m. preceding day; (11) 2:55 p. m. and 11:25 p. m. of preceding day.

2. What is the time 9 hr. 24 min. later than the following times: (1) 2:30 a. m.? (2) 7:45 a. m.? (3) 1:12 p. m. (4) 11:38 p. m.?

Find the time which is 15 hr. 40 min. earlier than the following times: (1) 6:15 a. m.; (2) 1:50 a. m.; (3) 9:10 p. m.; (4) 2:15 p.m.

Longitude is reckoned east or west from an established meredian. The Meridian of Greenwich is used in the problems in this book, that being the one commonly so used.





4. Find the difference in longitude between the following places: (1) 15° W. and 75° W.; (2) 15° E. and 75° W.; (3) 30° E. and 45° E.; (4) 30° E. and 45° W.

5. Find the longitude of a person who has traveled as follows:

(1) Started at 45° E. and traveled 60° eastward;

(2) Started at 45° E. and traveled 70° westward;

(3) Started at 23° 45′ W. and traveled 65° eastward;

(4) Started at 35° W. and traveled 20° 30′ eastward;

(5) Started at 26° 27′ W. and traveled 38° 50′ westward.

Places on the earth cannot have more than 180° E. or W. longitude.

6. The longitude of San Francisco is 122° W., nearly. Find the longitude of a person who starts at San Francisco (1) 30° W.; (2) 65° E.; (3) 150° E.; (4) 75° E.

7. If one should start at Calcutta (88° E.) and travel 125° eastward, what would then be his longitude?

The time of day is determined by the revolution of the earth on its axis, the date and the day of the week are fixed by man. In order to avoid confusion of date the nations have agreed upon an International Date Line. This line runs along 180° E. or W. except where the latter runs across land, or would separate islands of a group. In such cases the date line runs east or west of 180°. The new day first begins at midnight on the International Date Line.

The time of any place in E. longitude is always later than that of any place in W. longitude.

When the difference in longitude in two places is found to be greater than 180°, it is not best to subtract that amount from 360° before finding the difference in time.

To avoid confusion in time tables and guard against accidents, the railway managers of North America have agreed on a system of time-keeping, called Standard Time. Meridians are chosen fifteen degrees apart, and all places near one of these meridians keep the time of that one. It follows that places near different meridians will keep times differing by one or more hours. The minute hands of all clocks keeping Standard Time will agree.

Eastern Standard Time is the time of 75° W.; Central Standard Time, the time of 90° W.; Mountain Time, of 105° W.; Pacific Standard Time, of 120° W.

8. How much does each of these times differ from the time of Greenwich?

9. Washington is 77° W. How much does the Stadard Time differ from its local time?

10. What is the difference between Eastern Time and Pacific Time?

11. Boston is 71° W. Is Standard Time too fast or too slow for Boston, and how much?

12. When it is 1 hr. 20 min. p.m. Saturday, at 110° E., at what place is 9 p. m.? What is the day at the latter place? Ans., Friday, 135° W.

13. What is the date and hour at S. F., 122° W., when it is 8 a. m., May 1st, at Manila, 121° E.? Ans., 3:48 p. m., April 30.

14. What is the date and hour at Pekin, 116° E., when it is 11:40 a. m., Oct., 20th, at S. F., 122° W.? Ans., 3:32 a. m., Oct. 21.

15. What is the time at Calcutta, 88° E., when it is 2 hr. 25 min. p. m. at St. Paul, 93° W.? Ans., 2:29 a. m., next day.

16. A ship which carries S. F. (122° W.) time, finds the local time to be 3 hr. 45 m. p. m., when the S. F. time is 10 hr. 20 m. a. m. What is the longitude of the ship? Ans., 40° 45′ W.

17. At another time the local time is found to be 8 hr. 27 m. a. m. when the S. F. time is 2 hr. 15 m. p. m. Where is the ship? Ans., 151° E.

18. Where is the ship when the local time is 4 hr. 45 m. p. m., when the S. F. time is 6 hr. 30 m. a. m.? Ans., 31° 45′ E.

19. If a ship carries Greenwich time, what is its longitude when the local time is 5 hr. 35 m. p. m. and the Greenwich time is 2 hr. 50 m. a. m.? If it is Monday at Greenwich, what is the day where the ship is situated? Ans., 138° 45′ W.; Sunday.

20. When it is 9:00 p. m. on Friday at 110° W. it is 2:20 p. m. at another place. What is the longitude of the latter place? What is the day of the week? Ans., 150° E., Saturday.

21. When it is 2:20 p. m. Sunday at 120° E. longitude, it is 8:05 p. m. at another place. What is the longitude of the latter place, and what day of the week is it? Ans., 153° 45′ W.; Saturday.

22. The exact local time at Lick Observatory is 8 hr. 6 min. 31.85 sec. slower than Greenwich time. What is the longitude of the Observatory? Ans., 121° 37′ 57.75″ W.

23. When it is 1:35 p. m. Saturday at 110° E., what is the hour and day at 110° W.? Ans., 10:55 p. m. Friday.

24. A watch is set right at Pekin, 116° 30′ E. On what meridian is the watch when at noon it shows 9:20 a. m.? Ans., 156° 30′ E.

25. When it is 5 minutes after 4 o'clock on Sunday morning at Honolulu, longitude 157° 52′ W., what is the time at Sidney, Australia, longitude 151° 11′ E. Ans., 41 m. 12 sec. a. m.

26. When it is 10 o'clock p. m. Wednesday, at longitude 20° E., what is the time at San Francisco, 122° W.? Ans., 12:32 p. m.

27. The time which is telegraphed over the state from Mt. Hamilton is Pacific Standard Time. What is the difference between this and the exact time of the Observatory?

Sawing Wood, Cutting Ice, Etc.

It is supposed in these problems that the charge is in proportion to the amount of work done.

1. If it costs $12 to survey the sides of a square forty acre field, how much will it cost to divide it into square lots of two and a half acres each? Ans., $18.

2. It cost $510 to fence a field 50 rods by 70 rods. How much additional will it cost to divide the field into lots 10 rods square? Ans., $1232.50.

3. If it costs $336 to fence a field 36 rods by 60 rods, how much additional will it cost to fence it when divided into the largest possible equal square lots? Ans., $462.

4. It cost $1050 to build a solid board fence about a lot 600 feet by 800 feet. How much additional should be paid to fence it into lots 200 feet by 150 feet? Ans., $1575.

5. If 60 cts. a cord is charged for sawing 4 ft. wood into 16 in. sticks, how much should be charged for sawing the same wood into 12 in. sticks? Ans., 90 cts.

The expense should be in proportion to the amount of sawing, and not to the number of sticks into which each stick of cordwood is cut. A diagram will be found helpful. For problems like the sixth, note that a stick of eight foot wood contains twice as much wood as a stick of four foot wood.

6. If 75 cts. a cord is charged for sawing 4 ft. wood into 12 in. sticks, how much a cord should be charged for sawing 8 ft. wood into 16 in. sticks? Ans., 62½ cts.

7. 75 cts. a cord is charged for sawing 4 ft. wood into 12 in. sticks, the charge including 25 cts. for a helper. If 30 cts. a cord is allowed for a helper, how much should be charged for sawing 8 ft. wood into 16 in. sticks? Ans., 71⅔ cts.

8. If 50 cts. a cord is charged for sawing 2 ft. wood into sticks 12 ins. long, how much should be charged

for sawing 5 cords of 4 ft. wood into sticks of the same length? Ans., $3.75.

9. If 7 cts. is paid for sawing a block of ice, $9\times10\times1$ ft., into pieces $2\times3\times1$ ft., what should be charged for sawing a block $9\times12\times1$ ft. into pieces $3\times4\times1$ ft.? Ans., $5\frac{1}{4}$ cts.

10. If 6 cts. is charged for sawing a block of ice $9\times10\times1$ ft. into pieces $3\times5\times1$ ft., how much should be charged for sawing a block $12\times15\times1$ ft. into pieces $3\times6\times1$ ft.? Ans., $13\frac{1}{29}$ cts.

11. If 11 cts. is charged for sawing a block of ice $20'\times30'\times2'$ into pieces $4'\times5'\times2'$, how much should be charged for sawing a block $15'$ $24'\times3'$ into pieces $3'\times6'\times3'$?

Ans., $10\frac{23}{40}$ cts.

12. If 24 cts. is charged for painting a cubic foot of wood on the outside, how much additional should be charged for painting it when divided into pieces $2\times3\times4$ in.?

Ans., 80 cts.

13. $13.05 is paid for polishing a block of stone $8\times10\times12$ ft. How much additional should be paid for polishing the same when divided into pieces $3\times4\times5$ ft.? Ans., $20.10.

14. If it takes 72 hours to lath a room $48\times60\times15$ ft., how long will it take when it is divided into rooms $12\times16\times15$ ft.? Ans., $182\frac{2}{17}$ hrs.

Working Problems.

When money is to be paid, divide it among the workers in proportion to the amount of work done.

1. A and B together can do a piece of work in 15 days. After working together 6 days, A leaves and B finishes the work in 30 days more. In how many days can each alone do the work?

A, $21\frac{3}{7}$ da.; B, 50 da.

2. A and B together can do a piece of work in 12 days. After working together for 9 days, however, they call in C to help them, and the three finish the work in 2 days. In how many days can C alone do the work? Ans., 24 da.

3. Henry can do a certain piece of work in 18 hours, John can do the same in 12 hours, and their father in 6 hours. Henry begins work at 7 o'clock, John begins at 8, and their father is to begin in time for the work to be finished by noon. When must their father begin work?

Ans., 9:40.

4. A can do a piece of work in 20 days, A and B in 12 days, B and C in 10 days. In what time can C do the work alone? Ans., 15 da.

5. Three men, 4 women, or 5 boys can do a job of work in 8 hours; in what time can 1 man, 2 women, and 3 boys do it working together? Ans., $5\frac{25}{43}$ hr.

6. One pipe can fill a tank in 8 hours, a second in 11 hours, a third can empty it in 15 hours. If the tank is empty and the pipes are all opened, in what time will the tank be filled? Ans., $6\frac{138}{197}$ hr.

7. A, B and C can do a job of work in 10, 12, and 15 days respectively. A works 4 days, B 3 days, and C finshes the work. If $30 is paid for the work, how much should each receive?

Ans., A, $12; B, $7.50; C, $10.50.

8. A, B and C together can do a piece of work in 10 days; A and B together in 12 days; B and C together in 20 days. How long will it take each alone to do the work?

9. Henry and Samuel could have done a piece of work in 15 hours, but after working together for 6 hours, Samuel was left to finish it, which he did in 30 hours. In what time could Henry have finished the work if Samuel had left at the end of 6 hours?

10. A can dig a well in 8 days, and B in 12 days. They work at it on alternating days, A beginning. How long will it take to dig the well? If $24 is paid for digging the

well, how much should each receive?

Ans., $9\frac{1}{2}$ days; A receives \$15; B, \$9.

It is necessary in problems like the tenth to find the period which is repeated and deal with it as a whole. A does one-eighth of the work in one day, and B one-twelfth, hence they do together five-twentyfourths in two days. Since they work on alternate days, the repeating period is two days. It will take four whole periods and there will remain four-twentyfourths. On the day following the fourth period A does three-twentyfourths, leaving one-twenty-fourth for B to finish. He can do this in one-half a day.

11. A can do a job in 10 days, B in 12 days. If A works in the afternoons only, and B works all day, how long will it take to do the work? How much should each receive if \$15 is paid for the work?

Ans., $7\frac{7}{11}$ da.; A rec. $\$5\frac{5}{11}$; B rec. $\$9\frac{6}{11}$.

12. A can do a piece of work in 10 days, B in 12 days, A and B work on alternate forenoons, A beginning; both work in the afternoon. How long will it take to do the work? \$18 is paid for the work. How much should each receive? Ans., $7\frac{2}{5}$ da.

13. A man and a boy undertake a piece of work. The man alone can do the work in 8 days, and the boy in 12 If the man begins work on the first day and works every other day only, and the boy works every day from the first, how long will it take them to complete the work?

Ans., $6\frac{3}{5}$ da.

14. A can do a piece of work in 10 hours and B in 12. They begin the work together at 7 a. m. A works an hour and rests half an hour, and so continues. B works all the time. When will the work be completed? Ans., 1:35—p. m.

Traveling and Rowing.

1. Train No. 1, which travels 24 miles per hour, passes a stake in 15 sec. Train No. 2, which travels 30 miles per hour, passes a stake in 16 sec. How long is each train? How long will it take the trains to pass each other on a double track, if going in opposite directions? How long if going in the same direction?

Ans., length of trains, 528 ft., 704 ft.

2. It is 120 miles from Yolo to Milpitas. Train No. 1 (above) leaves Yolo at 7 a. m. to go to Milpitas and return, and stops 15 min. at Milpitas. Train No. 2 (above) leaves Yolo for Milpitas at 9 a. m. Where will trains meet?

Ans., 10 miles from Milpitas.

Find the location of Train No. 2 when Train No. 1 starts back from Milpitas. The first train reaches Milpitas at 12 m. and starts back at 12:15. The second train will be $97\frac{1}{2}$ miles from Yolo and $22\frac{1}{2}$ miles from Milpitas at that time. They now approach each other at a combined rate of 54 miles an hour and will meet in 25 minutes 10 miles from Milpitas.

In problems where there is a change of rate or a stop find the position of the parties after the last stop or change and then there will be little difficulty in completing the work.

3. The distance from San Jose to San Francisco is 51 miles. A can ride the distance in $4\frac{1}{4}$ hours, B in $5\frac{1}{10}$ hours. A leaves S. F. for San Jose at 8 o'clock; B leaves S. J. for S. F. at 9 o'clock. When will they meet?

Ans., 10:46 o'clock.

4. A and B start at 7 o'clock a. m. to travel over the mountains into San Joaquin valley. A travels 6 miles an hour up hill and 9 miles an hour down hill. B travels 5 miles an hour up hill and 10 miles an hour down hill. It is 20 miles to the top of the grade. When and where will they next be together?

Ans., 5 p. m.; 80 miles from San Jose.

5. If A can go from Albany to Boston in $9\frac{1}{2}$ hours, and B from Boston to Albany in $11\frac{1}{3}$ hours, and they start at the same time, in how many hours will they meet?

Ans., $5\frac{21}{125}$ hrs.

6. A train 600 feet long is traveling 25 miles an hour. How long will it take it to pass entirely through a tunnel 1800 feet long?

Ans., $1\frac{1}{11}$ min.

7. James starts from San Francisco, which is 50 miles away, at 8 a. m. He rides 8 miles an hour for 3 hours, is delayed one hour by an accident, and proceeds at 6 miles an hour. Silas starts at 9 a. m. and rides 7 miles an hour. Where will he pass James?

Ans., 42 miles from S. F.

8. A wagon leaves San Jose at 7 a. m. to go to the coast, traveling 5 miles an hour up hill and 8 miles an hour down hill. A carriage leaves at 8 a. m. and travels 6 miles an hour up hill and 10 miles an hour down hill. It is 20 miles to the top of the grade and 40 miles from the top to the point on the coast. Where will the carriage overtake the wagon?

Ans., $33\frac{1}{3}$ miles from San Jose.

9. A man can row 5 miles an hour in still water. How fast can he row against a current which runs 3 miles an hour? How fast can he row with the same current?

10. How long will it take the man to row up stream 12 miles and back?

11. How far can he row down stream and back in 5 hours?

12. A team can travel up hill 3 miles an hour and down hill 8 miles an hour. How far can the team travel up hill and back in 7 hours?

13. A man who travels by team 10 miles an hour, and on foot 4 miles an hour, has a journey of 36 miles to make. How far must he go by team that he may complete the journey in six hours from the time of starting? Ans., 20 mi.

14. A and B started from M and N respectively and traveled till they met, when it was learned that A had traveled five-sevenths as far as B. If A had traveled 18 miles farther, and B had traveled the same as at first, A would have traveled twice as far as B. How far did each travel? Ans., A 10 mi.; B 14 mi.

15. A starts to walk from P to Q at the rate of 4 miles an hour, and 1 hour later B starts from P and overtakes A in 4 hours. Walking on, B arrives at Q 2 hours before A. Find the distance from P to Q.

Ans., 60 mi.

16. It is 11 miles from San Jose to Los Gatos A carry-all leaves San Jose at 8 o'clock to go to Los Gatos and return and travels 5 miles an hour. When must a carriage which travels 8 miles an hour leave San Jose that it may meet the carry-all and return to San Jose by 12 o'clock? Ans., 10:40.

17. It is 24 miles from San Jose to the Lick Observatory. A carriage leaves San Jose at 9 a. m. and travels up hill at the rate of 4 miles an hour, stops an hour at the observatory, and returns at the rate of 10 miles an hour. A second carriage leaves San Jose at 3 p. m. and travels at the rate of 5 miles an hour. How far from San Jose will the carriages pass each other?

Ans., $11\frac{1}{3}$ mi.

18. A man can row up stream 3 miles an hour and down the stream 7 miles an hour. How far can he row up stream and back in 5 hours?

Ans., $10\frac{1}{2}$ mi.

19. A started at 8 a. m. around a mile track, walking at the rate of 5 miles an hour B started at 8:05 and walked in the same direction at the rate of 4 miles per hour. If A rests one minute at the end of each mile, when will they first be together again? Ans., 9 o'clock.